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MISCELLANY

OF

MATHEMATICAL PROBLEMS.

In Three VOLUMES.

By **ANTHONY THACKER**,
TEACHER of the MATHEMATICKS at
Birmingham FREE-SCHOOL.
And the AUTHOR of the LADIES DIARY.

VOL. I. CONTAINING,

- I. A New Method of solving *Geometrical Problems*.
 - II. A Complete *Treatise of Spherical Trigonometry*, wherein each Case is solved as well by a neat Equation, to lay a Foundation for solving difficult *Problems in Spherics*, as by the *Logarithms* for common Purposes.
 - III. A COLLECTION of *Spherical Problems*.
 - IV. Contains a Variety of *unlimited Questions*, and *Diophantine Problems*.
 - V. The *Solutions* to the Questions in the *Gentleman's and Ladies Diary*, for the present Year 1743.
-

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Collection of Special Reports

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The solution to the Questions in the Questions

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THE UNIVERSITY OF CHICAGO

1. The first part of the document is a list of names and titles, including "The Hon. Mr. Justice" and "The Hon. Mr. Justice".



P R E F A C E.

 *N* an Age so plentifully abounding with Books upon all Subjects, as this, and wherein all Themes upon which the Pen can well be employed, seem in a Manner to be exhausted, it must be extremely natural to a Person who would not be imposed on by magnified Pretensions and specious Title Pages, to enquire concerning a new Book;

What Discoveries and Improvements it contains in the Subject it handles?

Whether it serves the World by detecting any Errors or false Principles, which have been made or adhered to by such as have before undertaken to write upon the same Thing? Or,

Whether it be helpful to Learners, and agreeable to Men of Knowledge, upon account of placing Things before discovered and known, in a new and more advantageous Light; whereby the Learning of them may be facilitated, and their Knowledge diffused thro' a greater Sphere?

As every Lover of Books, who is at the same Time a Person of Sense, can only be induced to purchase any particular Book from an Affirmative Answer to one or more such Queries as these; an Author must have the greatest Presumption, who should offer a Piece

to the World void of all these Qualifications, in a Manner to demand its Patronage, for what he is conscious to himself has no Claim to the Publick Acceptance.

As to this Piece of ours, we were inspired with the Confidence of publishing it, from this Motive, that we presumed it would contain some Things new, and hitherto unobserved, which were of considerable Use, and tended to the Improvement of the Analytic Art, by introducing some new Methods of Argumentation, which usually produce more elegant Conclusions than the common Ones in Geometrical and Spherical Problems; and those of all Kinds in the Solutions of which, Sines and Tangents can any Way be employed.

But not to talk only in the general, we shall here present the Reader with a brief Account of the several Things contained in this Volume, in the Order they lie, whereby he may be the better enabled to form a Judgment thereof.

I. The First is a Problem taken from Sir Isaac Newton's Universal Arithmetic, from our Solution of which are deduced the two Theorems of Ptolomy, for finding the Sine of the Sum, and Sine of the Difference of two Arches; the two Theorems for finding the Cosine of the Sum, and Cosine of the Difference of two Arches: And afterwards, by Way of Corollary, four other Theorems, for finding in Terms of Tangents, the Tangents and Cotangents of the Sums and Differences of Arches; all of excellent Use in the Solution of difficult Problems, as by the following Pages will evidently appear. We made Choice of this Problem, from the Solution of which to investigate these Theorems, in order to free us from the greater Trouble of giving to each a Synthetical Demonstration: For notwithstanding they are all extant in other Authors, yet as they are made the Foundation of most of our Argument through the Book, it would not have

P R E F A C E.

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have been so well to have let them passed without some Manifestation of their Truth.

2. The 2d, 3d, 4th, 5th and 6th Problems are either taken from Newton, or to be found in common Authors, but solved in a different Manner to what they are in the Books from whence they were taken, and so as to point out a Collection of curious and elegant Theorems (vide p. 1. 13, and 14.) of good Use in the Solution of most Problems that can be proposed about a plain Triangle, and that too frequently without the Help of Algebra.

3. The 7th is a common Case of plain Trigonometry, but proposed here in order to introduce the Investigation of two elegant Theorems, which may indeed be used with less Advantage than the common Ones, in the Management of Numbers, but serve to many valuable Purposes in Algebraical Calculations. For such is the Difference between Arithmetick and Algebra, that, in many Cases, those Theorems which are most commodious in the Management of Numbers, incumber Algebraical Equations with superfluous Difficulties; and that, on the contrary, those which most promote Algebraical Calculus, frequently protract the Length of an Arithmetical Supputation: Instances whereof may be seen in the two Theorems we are speaking of, and in several belonging to Spherical Trigonometry.

4. Next follow ten Examples shewing the Use of the Theorems in Pages 13 and 14.

5. The 8th, 9th, and 10th Problems are taken from Sir Isaac Newton's Universal Arithmetick, with this Alteration, that whereas Sir Isaac only solved the Problems by supposing the Angle opposite to the Base or longest Side to be $= 90$ Degrees, we have solved them when the Angle is 90 , or any other Number of Degrees.

6. The 11th and 12th are new Problems, and point out a Method of finding the Value of x in the Equa-

tion $Axx + Bx\sqrt{1-xx} = C$, by a Quadratic Equation.

The 13th is a curious Problem by Mr. Tho. Bird.

8. The 14th, 15th, 16th, 17th and 18th Problems are such as we hope will be well received, because they point out a Method of solving Variety of Problems, in Ward and other Authors, by some easy Analogies, which those Authors have done by Quadratic Equations.

9. The 19th is from Mr. S. D. the 20th from Sir Isaac Newton, the 21st by Mr. Robert Heath, and the 22d a very pretty Problem in Altimetria proposed by Mr. Henry Travis, in Ladies Diary 1740.

10. The latter Part of Problem 23d, which requires the Tangent and Cotangent of double an Arch whose Tangent is given, is extant in other Authors; but the former Part thereof, which requires the Sine and Cosine of double a given Arch, in Terms of the Tangent of the said given Arch, is new, at least to

us, and the Expressions $\frac{2t}{1+t^2} = x$ and $\frac{1-t^2}{1+t^2} = y$ of

immense Use in keeping Equations void of the Incumbrance of Surd Quantities, and making the Involution of the known Quantities unnecessary, as the Solutions to several of the following Problems do amply demonstrate.

11. Prob. 24. is from Ward, and was also proposed as a Prize Question in the Ladies Diary 1725. Prob. 25. was proposed by Mr. W. Deare, in Diary 1720. The 26th is taken from Ward; the 27th is the 26th in more general Terms, by Help of the Solution to which, the 28th, taken also from Ward, is very easily solved. The 29th is a difficult Problem contained in Ward, and also proposed for a Prize Question in the Diary for 1723. The 30th may be found in the first Edition of Ronayne's Algebra; where it is said to have busied the ablest Mathematicians in Europe; and was also a Prize Question in
the

the Diary of 1727. The 31st is by Mr. Robert Fearnside of Hull. The 32d is a very pleasant Problem by Mr. John Watts of Comb; the 33d is a curious Problem by Mr. John Turner of York, and the 34th is another of the same Nature, by Mr. John Ladd of Cossal. Several of these Problems are solved by the Expressions deduced from the Solution to the above-mentioned Prob. 23. and all of them in such a Manner as we hope will be acceptable.

12. Supposing the Problems before nominated sufficient to lead the Readers into a Tract of Reasoning, by which he may be enabled to give the most concise and elegant Solution to any Thing which can ordinarily be proposed, in plain Geometry and right-lined Trigonometry; we proceed next to Spherics, of which we have first laid down the Elements, consisting of Definitions and Principles, in the most plain and familiar Manner we could devise, giving a Demonstration to each of the more weighty and important Principles; which being finished, we next lay down and demonstrate the two common Theorems used in the Solution of right-angled Spherical Triangles, from which we draw the four others necessary to the same Purpose; and then after giving some Direction and Advice about the Algebraical Manner of Solution, and laying down our new Method of multiplying Sines and Cosines, together with its Grounds and Reasons, we proceed to the actual Solution of the several Cases of right-angled Triangles, which we give both by the Logarithms, in the common Form, for the Service of such as would make a practical Use of this Kind of Knowledge, and by an Algebraical Equation, for their Purpose who would lay a Foundation for solving difficult Problems in Spherics. These being finished, and a Synopsis afterwards given from the preceding Pages, exhibiting at one View the Analogy and Equation for each Case, the Theory of oblique-angled Spherical Triangles is undertaken;

undertaken; in the Variety of Theorems contained in which, we hope the Reader won't miss of some Satisfaction and Entertainment, many of them being of wonderful Use, in giving the most general and concise Solutions to all Kind of Spheric Problems; and, we humbly presume, all of some Subsistence to the same End. The Reader will observe, by casting his Eye over the whole Class of our Spherical Theorems, that some certain among them, are of an admirable and extensive Nature, so that by them alone, with the Assistance of Algebra, properly to reduce and adapt them to the Case in Hand, the whole Art may be performed; and the Solution of one Case being thus drawn, by Way of Corollary from that of another, the Compass wherein the whole Science is contained vastly contracted. In the first Scheme of this Work we had handled the Trigonometry according to this Method, which gave occasion to an Advertisement in the Ladies Diary, that the whole would be performed by only Six Theorems; but we have since been persuaded, by some Friends who are solicitous of having it levelled to all reasonable Capacities, to change its former Order into that in which it at present appears: To our own Disadvantage indeed, in augmenting the Quantity of Paper when the Price of the Book was fixed; but we hope, neither to the Disadvantage or Displeasure of either our learned or less skilful Readers: Not of the former, since not one of those Theorems which were advertised in the Diary is suppressed, but all inserted in their proper Places, with a gratis Addition of many more; not of the latter, because no Body can deny but the present Form is more beneficial and useful to them, than that which scarce moved a Step without Algebra.

As in the right, so in the oblique-angled Part of Spherical Trigonometry, no Case is gone over without at least two Solutions, one algebraical, and the other

other by *Logarithms*: And several Cases in each Part have an additional One given to them by our new *Sinical Multiplication*; a Method which would have been highly valued before the Invention of *Logarithms*, and can't now be denied to be curious. *To each Case in both Parts, the Solution of its Ambiguities is subjoined.*

13. Next follow some *Spherical Problems*, taken from the *Ladies Diary*, *Petersbourg Acts*, and *Gentleman's Magazine*, (with their *Solutions* from our *Theorems*) not many in Number, but yet sufficient to initiate an ingenious Person into such a Manner of arguing, as will carry him through Problems of this Kind with great Facility, especially with the Addition of those of this Sort which are intended to be inserted in the *Third Volume*.

14. These Problems are succeeded by some of the unlimited Kind, and those again by some from *Diophantus*; all which are solved in the most easy and natural Manner we could devise.

15. Finally, the *Solutions* to the Questions contained in the *Gentleman's and Ladies Diaries* for the present Year compose the last Part of this Volume.

Thus having finished the *Narrative Part* of this Work, we shall only add, that in *Triangles* similar to one another,

1. As the *Rectangle* of any two Sides of one Triangle: is to its Area :: so is the *Rectangle* of the two answerable Sides in another Triangle: to its Area.

2. As the *Square* of the Sum of any two Sides: is to the Area :: so is the *Square* of the Sum of the two corresponding Sides in another Triangle: to its Area.

3. As the *Square* of the Difference of any two Sides: is to the Area :: so is the *Square* of the Difference of the like Sides in another Triangle: to its Area.

4. As

4. As the Square of the Perimeter of one Triangle : is to its Area : : so is the Square of the Perimeter of another : to its Area.

5. Among the Spheric Theorems this (serving to the same Purposes as Theorem 9.) was designed a Place, viz.

Multiply the Rectangle of two Sides of a Spherical Triangle, by the versed Sine of their contained Angle ; the last Product subtracted from the Cosine of the Difference of the said Sides, shall leave the Cosine of the third Side.

N. B. When the last mentioned Product can't be taken from the Cosine of the Difference of the Sides, take the said Cosine from it ; the Remainder being negative, shews the third Side, in such Circumstances, to be obtuse. (See the Reasoning at p. 75. and 76.)

E R R A T A in Vol I.

P A G E 6. Line 10. for Tangent read Cotangent. p. 171. l. 28. r. as $796 : 50 + 10 :: 50 - 10 : .753768$ &c = the Cotangent of half the Sum of the Angles at the Base = $52^\circ : 59' : 34''$ and per Theorem 10. p. 13. as $50 : 10 :: \text{Sine } 52^\circ : 59' : 34''$ to half the Difference of the Angles at the Base = $9^\circ : 11' : 25''$. Hence the lesser Angle at the Base is $43^\circ : 28' : 09''$ — p. 53. l. 2. for *Mn*, r. *MN*. l. 15. r. *2BN*. and in the same l. after 90° read *BN*, in l. 16. and 17. r. *BN*. in l. 18. and 19. r. *BRN*. p. 59. l. 25. for Circles r. Parallels. l. 26. for Circles r. Parallels. l. 27. for Secondaries r. Parallels to. p. 71. l. 26. after for *r*. in, and l. 20. for *CF* r. *BF*. p. 75. l. 25. for Question r. Quæsitum. p. 87. l. the last, for Cosine r. Cotangent. p. 124. l. 10. after in, r. the. l. 23. for Ang. *OVP* r. Ang. *VOP*. p. 133. l. 7. for only the, r. the only. p. 162. l. 9. after consequently *dele* that. p. 169. l. 28. for $80 - 3x$ r. $80 - y$. p. 170. l. 4. instead of makes, r. it makes, and in the same l. between x and 40 put =. l. 11. for 3, 3, r. 2, 3. p. 171. l. 9. for Contradiction r. Contradistinction. p. 174. l. last for second r. 2d. p. 177. l. 7. instead of for, r. so for. p. 181. l. 11. for more, r. greater.

Note, The Angle opposite to the Base should have been mentioned in Problems the 8th, 9th, and 10th, as Part of the Data ; and if *t* be put for the Cotangent of its half, then the Equation at p. 23. l. 6. will become $\sqrt{S^2 + t^2 p^2} - tp = \text{the Base } EF.$ and that at l. 3. from the Bottom,

$$\frac{tm - \sqrt{t^2 m^2 - 2 t S_2 + S^2}}{2t} = B$$

ALGEBRA

Applied to the SOLUTION of

Some PROBLEMS

IN

PLAIN TRIGONOMETRY.

The Problems are generally taken from Sir ISAAC NEWTON, SAUNDERSON, and others, but solved in a different Manner.

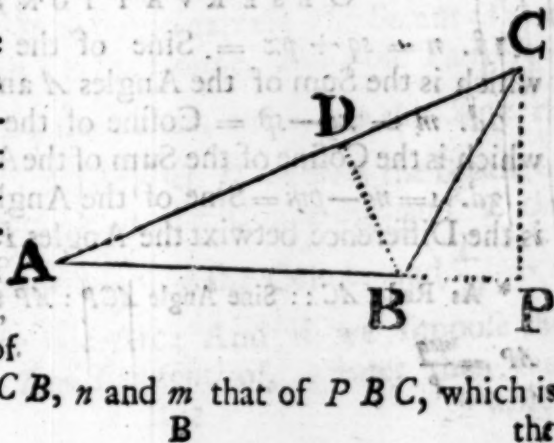
PROB. I.

Being the first of Sir Isaac Newton's Geometrical Problems, in his Universal Arithmetick, pag. 101.

Having all the Angles and Side BC of the Triangle ABC annexed given, to find the Perpendicular DB, &c.

SOLUTION.

PRODUCE the Side AB till it meets the Perpendicular CP, put p and q to express the Sine and Cosine of the Angle A , s and z that of the Angle ACB , n and m that of PBC , which is the



(2)

the Sum of the Angles BAC and BCA , and let
Rad. = 1, then *per* Trigonometry we have as $1 : e$

(= BC) :: $s : DB = se$, and as $p : DB = se :: 1 : AB$

= $\frac{se}{p}$. Again as $p : DB = se :: q : AD = \frac{seq}{p}$; as

$1 : e :: z : DC = ez$, consequently $\frac{seq}{p} + ez = AC$,

but $p : e :: n : AC = \frac{ne}{p} = \frac{seq}{p} + ez$, which multiplied
out of Fractions, &c. gives $n = sq + pz$. Again as

$1 : e :: m : BP = me :: me + \frac{se}{p} = \frac{neq}{p}$ * out of Frac-

tions it's $mp + s = nq :: s = nq - mp$; also from
 $n = sq + pz$ and $s = nq - pm$ by writing $nq - pm$ for
 s , we have $n = nq^2 - pmq + pz :: n - nq^2 = -pmq +$
 pz , or $n \times 1 - q^2 = pz - pmq$: But $1 - q^2$ is $= p^2 ::$
 $np^2 = pz - pmq :: np = z - mq :: np + mq = z$ and by
writing $sq + pz$ for n , we get $s = sq^2 + pzq - pm$ or
 $sp^2 = pzq - pm$, which $\div$ ed by p , gives $sp = zq - m$
 $:: m = zq - sp$. Hence we have the following The-
orems :

Theorem 1. $n = sq + pz$.

Theorem 2. $m = zq - sp$.

Theorem 3. $s = nq - pm$.

Theorem 4. $z = mq + np$.

OBSERVATIONS.

1st. $n = sq + pz$ = Sine of the Angle PBC ,
which is the Sum of the Angles A and C .

2^d. $m = zq - sp$ = Cosine of the Angle PBC ,
which is the Cosine of the Sum of the Angles A and C .

3^d. $s = nq - pm$ = Sine of the Angle ACB , which
is the Difference betwixt the Angles PBC and BAC .

* As Rad : $AC ::$ Sine Angle $ACP : AP$ that is $1 : \frac{ne}{p} :: q :$

$$AP = \frac{neq}{p}$$

4th. $z = mq + np$ = Cosine of the Angle ACB , which is the Difference betwixt the Angle PBC and BAC .

Which THEOREMS solve the following PROBLEMS.

I. To find the Sine corresponding to the Sum of any two Arcs or Angles.

THEOREM I.

The Sine of the Greater into the Cosine of the Less, *Plus* the Sine of the Lesser into the Cosine of the Greater.

II. To find the Cosine corresponding to the Sum of two Arcs or Angles.

THEOREM II.

The Product of the two Cosines, *Minus* the Product of the two Sines.

III. To find the Sine corresponding to the Difference of two Arcs.

THEOREM III.

The Sine of the Greater into the Cosine of the Less, *Minus* the Sine of the Less into the Cosine of the Greater.

IV. To find the Cosine corresponding to the Difference of the two Arcs.

THEOREM IV.

The Product of the two Cosines, *Plus* the Product of the two Sines.

COROLLARY.

Whence if B be put to express the Secant of that Arc, of which q is the Cosine, and t the Tangent; then will $\frac{1}{B} = q$ the Cosine, and $\frac{t}{B} = p$ the Sine of the said Arc. Again, let D express the Secant of that Arc, which z is the Cosine of, and T the Tangent; then will $\frac{1}{D} = z$ the Cosine, and $\frac{T}{D} = s$ the Sine of the said Arc: And if we suppose the Arc which T is Tangent of, greater than that

which t is the Tangent of: then by the 1st and 2d foregoing Theorems we have $\frac{T}{BD} + \frac{t}{BD} = \text{Sine}$ of the Sum of the said Arcs, and $\frac{1}{BD} - \frac{Tt}{BD} = \text{the Co-}$
 sine of their Sum; whence *Theorem 5.* $\frac{T+t}{1-Tt} =$
 Tangent of the Sum of the said Arcs; and *Theo-*
rem 6. $\frac{1-Tt}{T+t} = \text{Cotangent of their Sum.}$ Again,
 by the 3d and 4th of the foregoing Theorems, we
 have $\frac{T}{DB} - \frac{t}{DB} = \text{Sine of the Difference of the said}$
 two Arcs, and $\frac{1}{DB} + \frac{Tt}{DB} = \text{Cosine of their Diffe-}$
 rence: Hence *Theorem 7.* $\frac{T-t}{1+Tt} = \text{Tangent of}$
 the Difference of the said Arcs; and *Theorem 8.*
 $\frac{1+Tt}{T-t} = \text{Cotangent of their Difference.}$

Note, The greater any Sine n is, the less will its Cosine m be; and when n is the Sine of 90 Degrees, then its Cosine m is $= 0$: And when the Sine n is the Sine of a greater Arc than 90 Degrees, then m is a negative Number, and must be expressed thus $-m$; hence the Letters which represent Cosines in the above Theorems, must sometimes be expressed by a negative Sign; and by that means the Signs in the said Theorems are altered, as will appear in the following Problems.

PROB.

P R O B. II.

Being the 5th in Sir Isaac Newton's *Algebra*,
pag. 104.

Having given the *Altitude*, *Base*, and *Sum of the Sides of any Triangle*, to find the *Angles*, &c.

PUT B for the Base EF ,
 p for the Perpendicular GH , and
 S for the Sum of the Sides EGE .
 GE , and let s



and q be put to express the Sine and Cosine of half the Sum of the Angles at the Base, x and y the Sine and Cosine of half their Difference; Radius = Unity. Then per *Trigonometrical Theorems*, p. 3. $sy + qx$ and $qy - sx$ will express the Sine and Cosine of the greater Angle at the Base (as the Angle at F) and $sy - qx$ and $qy + sx$ that of the Angle E , and per *Trigonometry*, as the Sine of the Angle E :

$GH :: \text{Radius} : EG = \frac{p}{sy - qx}$. And as the Sine of the Angle F : $GH :: \text{Radius} : GF = \frac{p}{sy + qx}$.

Whence $EG + GF$ is $\frac{p}{sy - qx} + \frac{p}{sy + qx} = S$, or when reduced to a common Denominator, &c. it is

$\frac{2psy}{s^2y^2 - q^2x^2} = S$, which multiplied by B , gives $\frac{2Bpsy}{s^2y^2 - q^2x^2} = BS$. Again as the Sine of the Angle F : $EG :: \text{Sine}$

Angle EGF : EF , that is $sy + qx : \frac{p}{sy - qx} :: 2sq :$

$\frac{2sqp}{s^2y^2 - q^2x^2} = EF = B$, which multiplied by S , gives $2sqpS$

$\frac{2spqS}{s^2y^2 - q^2x^2} = BS$, consequently $qS = By \cdot \frac{qS}{B} = y$, and
 $\frac{q^2S^2}{B^2} = y^2$: But as $\frac{2psy}{s^2y^2 - q^2x^2} = S \therefore 2psy = Ss^2y^2 - q^2x^2S$,
 or by writing $1 - y^2$ for x^2 it's $2psy = Ss^2y^2 - q^2x^2S$;
 but $Ss^2y^2 + q^2y^2S = Sy^2$ (because $s^2 + q^2 = 1$) $\therefore 2psy$
 $= Sy^2 - q^2S$, and by substituting for y and y^2 their
 Values, we have $\frac{2psqS}{B} = \frac{q^2S^2}{B^2} - q^2S$, which being di-
 vided by qS , gives $\frac{2ps}{B} = \frac{qS^2}{B^2} - q \therefore 2pB^2 = qS^2 - qB^2 \therefore$
 $\frac{q}{B} = \frac{S^2 - B^2}{2pB}$. But $2pB = 4A$.* Hence $\frac{q}{B} = \frac{S^2 - B^2}{4A}$ the
 Tangent of half the Sum of the Angles (E and F)
 at the Base, or the Tangent of half the Angle oppo-
 site to the Base. And $\frac{qS}{B} = y$ the Cosine of half the
 Difference of the Angles at the Base. Q. E. I.

Note, If instead of the Sum of the Sides $EG + GF = S$, there had been given their Difference $= D$,
 then would $\frac{p}{sy - qx} - \frac{p}{sy + qx} = D$, which reduced
 to one Denominator, &c. will give $\frac{2qxp}{s^2y^2 - q^2x^2} = D$,
 which multiplied by B , gives $\frac{2qxpB}{s^2y^2 - q^2x^2} = DB$. But
 we have already proved $\frac{2sqp}{s^2y^2 - q^2x^2} = B$, which mul-
 tiplied by D , makes $\frac{2sqpD}{s^2y^2 - q^2x^2} = BD$, consequently
 $xB = sD \therefore x = \frac{sD}{B}$, and $x^2 = \frac{s^2D^2}{B^2}$. But as $\frac{2sqp}{s^2y^2 - q^2x^2}$
 $= B \therefore 2sqp = Bs^2y^2 - Bq^2x^2$, or by writing $1 - x^2$
 instead of y^2 , it's $2sqp = Bs^2 - Bs^2x^2 - Bq^2x^2$, or

* = Four Times the Area of the Triangle.

$sqp = B^2 - Bx^2$, and by writing $\frac{s^2 D^2}{B^2}$ instead of x^2 , it becomes $sqp = B^2 - \frac{s^2 D^2}{B}$ $\therefore 2qpB = B^3 - s^2 D^2$ $\therefore \frac{q}{s} = \frac{B^2 - D^2}{2pB}$, or because $2pB = 4A$ it's $\frac{q}{s} = \frac{B^2 - D^2}{4A}$ the Cotangent of half the Sum of the Angles E and F at the Base, or the Cotangent of half the Angle opposite to the Base; and $\frac{sD}{B}$ is the Sine of half the Difference of the Angles at the Base.

P R O B. III.

Suppose there be given all the Sides of the Triangle separately, to find the the Angles. See Sir Isaac Newton's Algebra, pag. 124.

THEN because $By = qS$, and $Bx = Ds$ $\therefore B^2 y^2 = q^2 S^2$, and $B^2 x^2 = D^2 s^2$, which two Equations added together makes $B^2 y^2 + B^2 x^2 = q^2 S^2 + D^2 s^2$, or rather $B^2 = q^2 S^2 + D^2 s^2$ (because $x^2 + y^2 = 1$) and by writing $1 - s^2$ for q^2 it's $B^2 = S^2 - S^2 s^2 + D^2 s^2$ $\therefore S^2 s^2 - D^2 s^2 = S^2 - B^2$, and $s^2 = \frac{S^2 - B^2}{S^2 - D^2}$ $\therefore s = \frac{\sqrt{S^2 - B^2}}{\sqrt{S^2 - D^2}}$ ($=v$) the versed Sine of the Sum of the Angles at the Base.

If the Area be required; then because it is proved at p. 6. Line 8. $\frac{s}{q} = \frac{S^2 - B^2}{4A}$ $\therefore 4A = S^2 - B^2 \times \frac{q}{s}$

But we have also proved at p. 7. l. 3. $\frac{q}{s} = \frac{B^2 - D^2}{4A}$

* See p. 6. l. 1. and 20th.

which

which put for $\frac{q}{s}$, gives $4A = S^2 - B^2 \times \frac{B^2 - D^2}{4A} \therefore 16AA = S^2 - B^2 \times B^2 - D^2 \therefore A = \frac{1}{4} \sqrt{S^2 - B^2 \times B^2 - D^2}$ the required Area,

P R O B. IV.

If there be given the Area, Perimeter, and one of the Angles, (as the Angle EGF) of any Triangle, to determine the rest. See Sir Isaac Newton's Algebra, pag. 103.

THEN because we have already proved that $\frac{p}{sy - qx} = EG$, and $\frac{p}{sy + qx} = GF$, and $2qs =$

Sine of the Angle EGF: Hence $\frac{p}{sy - qx} \times \frac{p}{sy + qx}$

$\times 2qs = 2A \therefore \frac{p^2 qs}{s^2 y^2 - q^2 x^2} = A$, or $\frac{p^2 qs}{y^2 - q^2} = A$. Put

Z for the Sum of all the Sides of the Triangle, then will $Z - B = S$, which being multiplied by q , gives $Zq - Bq (= Sq) = yB$, (see the Equation p.6. l. 1.) $\therefore$

$\frac{Zq}{y + q} = B \therefore \frac{2A}{B} = \frac{2A \times y + q}{Zq} = p$, which squared

and substituted for p in the Equation above, gives

$\frac{4A^2 \times y + q^2 \times q^2}{y^2 - q^2 \times Z^2 q^2} = A \therefore \frac{4sA \times y + q}{y - q \times qZ^2} = 1 \therefore 4Ays +$

$4Aqs = qZ^2 y - q^2 Z^2 \therefore 4Aqs + q^2 Z^2 = qZ^2 y - 4Ays$,

which divided by q , gives $4Aq \frac{s}{q} + qZ^2 = Z^2 y -$

$4Ay \frac{s}{q}$. Call $\frac{s}{q}$, t , then will $4Aqt + qZ^2 = Z^2 y -$

$4Ayt \therefore \frac{4Aqt + qZ^2}{Z^2 - 4At} = y$ the Cosine of half the

Difference of the Angles $\therefore \frac{Z^2 - 4At}{2Z} = B$ the Base.

P R O B,

P R O B. V.

In the Triangle EGF, (see Figure p. 5.) is given the Perimeter, or Sum of the Sides $EG + GF + EF = Z$, the Perpendicular $GH = p$; and the Angle opposite to the Base, to find the rest. See Newton's *Algebra*, p. 162. or Hammond's *Algebra*, p. 282.

SOLUTION.

BECAUSE it is found that $\frac{p^2qs}{y^2 - q^2} = A$ and $\frac{2Axy + q}{Zq} = p$. (See p. 8. l. 12, and 16.) we have

by substituting $\frac{p^2qs}{y^2 - q^2}$ for A ; $\frac{2p^2qsxy + q}{y^2 - q^2 \times Zq} = p$, hence by dividing p out of both Sides of the Equation, and dividing the Numerator and Denominator by $qxy + q$, we get $\frac{2ps}{y - q \times Z} = 1 \therefore 2ps = yZ - qZ \therefore$

$\frac{2ps + Zq}{Z} = y$. The Cosine of the Difference of the Angles at the Base; and by substituting for y above, we have $\frac{qZ^2}{2sp + 2qZ} = \frac{2A}{p} = B$ the Base EF of the required Triangle. Q. E. I.

If the Angle opposite to the Base $= 90^\circ$ then s and q are equal, and $\frac{qZ^2}{2sp + 2qZ} = B$ becomes

$\frac{Z^2}{2p + 2Z} = B$; which is the same Equation as that given by Sir Isaac Newton:

P R O B. VI.

Given the Perpendicular $GH = p$, the Base $EF = B$, and the Angle (EFG) opposite to the Base, to find the rest.

SOLUTION.

IT is proved at p. 7. l. 1. $2sqp = Bs^2 - Bx^2 \therefore Bx^2 = Bs^2 - 2sqp \therefore 2x^2 = \frac{2Bs^2 - 4sqp}{B}$ the versed Sine of the Difference of the Angle at the Base, which may be better express'd, because $2s^2 =$ the versed Sine of the Sum of the Angles at the Base, being call'd v , and $2sq$ the Sine of the given Angle call'd n ; $\frac{Bv - 2np}{B} = 2x^2$, or rather $v - \frac{2np}{B} = 2x^2$ will express the versed Sine of the Difference of the Angles at the Base. Q. E. I.

Note, If the Perpendicular had fallen without the Base, as Figure p. 1. then because in the *Theorem* ($v - \frac{2np}{B} = 2x^2$) there is no Quantities which represent Cosines, nor any other which can from Affirmative become Negative, or from Negative become Affirmative, the *Theorem* $v - \frac{2np}{B} = 2x^2$, will universally be the versed Sine of the Difference of the Angles at the Base.

Hence the *Theorem* will easily solve the *Problem* proposed by Mr. Mason in the *Ladies Diary* for 1732. which in the succeeding *Diary* is solved by an Equation of the 4th Power.

P R O B.

P R O B. VII.

In any plain Triangle, as EGF , there is given the Side $EG=a$, the Side $GF=e$, and the included Angle EGF , to find either of the other Angles E or F ; and also the Side EF .

S O L U T I O N.

PUT s and z to express the Sine and Cosine of the given Angle EGF , x and y to express the Sine and Cosine of the Angle E , then find the Sine corresponding to the Sum of the Angles FEG and EGF , by Theorem 1. p. 3, viz. $sy + zx$, which is Sine of the Angle F . And per Trigonometry as Sine of the Angle $F : EG ::$ Sine of the Angle $E : GF$, that is, as $sy + zx : a :: x : e$, and by multiplying Extremes and Means we get $esy + ezx$

$= ax \therefore esy = ax - ezx \therefore \frac{es}{a - ez} = \frac{x}{y}$ the Tangent of

the Angle E , or $\frac{a - ez}{es} = \frac{y}{x}$ = the Cotangent of the

Angle E . Again to find the Side EF , as the Sine of the Angle $E : GF ::$ Sine of the Angle $EGF : EF$,

that is, as $x : e :: s : \frac{es}{x} = B$ the Base $EF \therefore \frac{es}{B}$

$= x$ and $\frac{e^2 s^2}{B^2} = x^2$. But per 47. e. 1. $1 - x^2 = 1 -$

$\frac{e^2 s^2}{B^2} = \frac{B^2 - e^2 s^2}{B^2} = y^2$ (Radius being = 1) $\therefore \frac{B^2 - e^2 s^2}{e^2 s^2}$

$= \frac{y^2}{x^2}$, and per Equation $\frac{a - ez}{es} = \frac{y}{x} \therefore \frac{a^2 - 2aez + e^2 z^2}{e^2 s^2}$

$= \frac{y^2}{x^2}$ consequently $a^2 - 2aez + e^2 z^2 = B^2 - e^2 s^2 \therefore$

$a^2 - 2aez + e^2 z^2 + e^2 s^2 = B^2$, or rather $a^2 - 2aez + e^2$
C 2 = B

$=B^2$ (because $z^2 + s^2 = 1$) the Square of the required Base EF . Q. E. I.

Note, z , in the Theorem $\frac{es}{a-ez} = \frac{x}{y}$ will pass from Affirmative to Negative, if the Angle EGF become greater than 90 Degrees, and then it will be $\frac{es}{a+ez} = \frac{x}{y}$. If the Angle EGF be just 90 Degrees then $z=0$, and therefore $\frac{es}{a} = \frac{x}{y}$. If the Value of $\frac{es}{a-ez}$; should come out Negative, then it will denote the Angle F , to be greater than 90 Degrees.

And Lastly, If the Value of $\frac{es}{a-ez}$, come out infinitely great, that is if $a-ez=0$, then it will denote the Angle E to be = 90 Degrees.

Also z , in $a^2 - 2aez + e^2 = B^2$, will pass from Affirmative to Negative when the Angle EGF from being less than 90 Degrees becomes greater than 90. And then the Theorem will be $a^2 + 2aez + e^2 =$ the Square of the Base EF . Q. E. I.

But perhaps it may not be disagreeable to our Readers if we add another Method of finding the Angle E ; which is very easy, thus from p. 6. l. 1. and 20th, where it is proved $qS = By$ and $x B = sD$ we get $\frac{qS}{y} = B$ and $B = \frac{sD}{x}$ consequently $\frac{qS}{y} = \frac{sD}{x} \therefore$

$\frac{x}{y} = \frac{sD}{qS}$ or as $S : D :: \frac{s}{q} : \frac{x}{y}$ the Tangent of half the Difference of the Angles E and F . And also from the foregoing Equations may be had $q^2 S^2 + s^2 D^2 = B^2$, or $2q^2 S^2 + 2s^2 D^2 = 2B^2$, which may be better expressed: For $2s^2$ is the versed Sine of the Sum Angles E and F , which being called v , and $2q^2$ the versed Cosine of the Sum called c ; then is cS^2

$S^2 + vD^2 = 2B^2$ the double Square of the required Base. Q. E. I.

It may be observed, this last Method furnishes us with the same *Theorem* which is commonly made use of for finding the Angle E , which is shorter than that we found before ; but the former is more useful in the Solution of difficult Problems, as will appear further on.

Here follow some very useful Theorems extracted out of the foregoing.

THEOREM IX.

As the Base of any Triangle : the Sum of its other two Sides :: the Cosine of half the Sum of the Angles at the Base : the Cosine of half their Difference. (See p. 6. l. 1.)

THEOREM X.

As the Base of any Triangle : the Difference of the other two Sides :: the Sine of half the Sum of the Angles at the Base : the Sine of half their Difference. (See p. 6. l. 20.)

THEOREM XI.

As four times the Area of any plain Triangle : its Perimeter :: the Difference betwixt the Sum of its Sides and Base : the Cotangent of half the Angle opposite to the Base. (See p. 6. l. 8.)

THEOREM XII.

As four times the Area : the Base *Plus* the Difference of the other two Sides :: the Base *Minus* the Difference of the other two Sides : the Tangent of half the Angle opposite to the Base. (See p. 7. l. 4)

THE

THEOREM XIII.

As twice the Rectangle of the two Sides : their Sum *Minus* the Base into the Perimeter of the Triangle :: Radius (1) : the versed Sine of the Sum of the Angles at the Base. (See p. 7. l. 19.)

THEOREM XIV.

As the Perimeter of any Triangle : twice its Area :: the Cotangent of half the Angle opposite to the Base : a fourth Number, which fourth Number being taken from half the Perimeter, leaves the Base of the Triangle. (See p. 8. l. the last.)

THEOREM XV.

As the Perimeter of any Triangle : twice the Perpendicular :: the Cosine of Half the Angle opposite to the Base : a fourth Number, which being added to the said Cosine of half the Angle opposite to the Base gives the Cosine of Half the Difference of the Angles at the Base. (See p. 9. l. 15.)

THEOREM XVI.

As the Base of any Triangle : to twice the Perpendicular :: the Sine of the Angle opposite to the Base : a fourth Number, which being taken from the versed Sine of the Sum of the Angles at the Base, leaves the versed Sine of the Difference of the Angles at the Base. (See p. 10. l. 12.)

Thus having given the most useful *Theorems*, which flow from the Solution of the foregoing *Problems*, we proceed now to illustrate their Uses in the Solution of a Variety of *Problems* (which are taken from Sir Isaac Newton, Saunderson, Ward, the *Ladies Diary*, &c.) without the Help of *Algebra* ; that is, we shall shew those who are only acquainted with common Arithmetick, and a Table of natural and artificial Sines, Tangents, &c. how

to

to solve most Questions which are to be found in Mathematical Treatises.

EXAMPLES.

And First, In the Solution of Problem 4th in WARD's *Mathematicks*, p. 316.

Say per Theorem 9. p. 13. As $97 : 137 :: \sqrt{\frac{1}{2}}$ (the Cosine of half the Sum of the Angle at the Hypothenufe) : $\frac{137\sqrt{\frac{1}{2}}}{97} = \frac{127\sqrt{2}}{194} =$, 99870, &c.

which answers in the Tables of Natural Cosines to $2^\circ . 55' . 30''$. * Hence the Angles at the Hypothenufe are 45° Plus $2^\circ . 55' . 30'' = 47^\circ 55' . 30''$. and 45° Minus $2^\circ . 55' . 30'' = 42^\circ . 4' . 30''$. Now per Trigonometry, as Radius : H :: Sine of $42^\circ . 4' . 30'' : 65$ the Cathetus sought. (which is easily found by adding together the Logarithms of 97 and $42^\circ . 4' . 30''$, &c.) Hence the other Leg is 72. Q. E. I.

In like Manner may the 6th Problem of NEWTON's *Algebra* be solved ; for it will be as $AB : AC + BC ::$ Cosine of half the Angle C : Cosine of half the Difference betwixt the Angles A and B, which being known the Angles A and B will be known, and then as Sine Angle C : AB :: Sine Angle B : AC. Q. E. I.

Also by this Theorem we might easily solve the 7th Question in the *Gentleman's Diary*, for the present Year 1743. and the required Sides would come out 20 and 22,364 very near.

EXAMPLE II.

See Problem 5th, WARD, p. 317.

Say per Theorem 10. as $97 : 7 ::$ Sine $45^\circ : \text{Sine } 2^\circ . 55' . 30''$ which is half the Difference of the Angles at the Base (or Hypothenufe). Hence the Angles at the Hypothenufe are $47^\circ . 55' . 30''$. and

* Or better say, $97 : 137 :: \text{Cosine } 45^\circ : \text{Cosine } 2^\circ . 55' . 30''$
42°.

42°. 4' 30". And so the Legs may be found as in the last, to be 65 and 72. Q. E. I.

And with the same Ease might the Legs have been found if the given Angle had not been a right one.

EXAMPLE III.

See NEWTON's *Algebra*, p. 104. Because there is the Perpendicular and Base given, the Area is also given, which call A , and let $AB + BC + CA = Z$, and $BC + CA - AB = D$, then say *per Theorem* 11. p. 13. as $4A : Z :: D : \frac{DZ}{4A} =$ Cotangent of half the Angle C , which being known, we may by *Theorem* 9. p. 13. find the Angles B and C , and then the Sides sought. Q. E. I.

EXAMPLE IV.

Shewing the Use of *Theorem* 11. p. 13. in the Solution of *Problem* the 4th in NEWTON's, *Algebra*, p. 103. Call the Cotangent of half the given Angle t ; the Perimeter Z , the Area A , and the Side opposite to the given Angle x , then *per Theorem* 11. p. 13. as $4A : Z :: Z - 2x : t$. Hence $4At = Z^2 - 2xZ :: 2xZ = Z^2 - 4At :: x = \frac{Z^2 - 4At}{2Z} = \frac{Z}{2} - \frac{2At}{Z} = BC$ the required Base. Q. E. I.

Whoever considers the Trouble of the Solution given by Sir *Isaac Newton* to this Problem, will, I presume, be pleas'd with what I've done. And the Reader is to observe, that this Solution ought to have been made to *Problem* 4. p. 6. But when I put that down, I confess that I did not see this easy Method of solving it.

EXAMPLE V.

Shewing the Use of *Theorem* 11. p. 13. in the Solution of the 19th *Problem* in SAUNDERSON's *Algebra*, p. 536. Vol. 2. The Cotangent of half

half the given Angle will in this Question be = to 1. Put x for the Sum of the required Legs, for the Area put A , the Hypotenuse b , then *per Theorem* $4A : x + b :: x - b : 1$. Hence $x^2 - b^2 = 4A$ and $x = \sqrt{4A + b^2}$ = the Sum of Legs, which being known the rest may easily be found by the foregoing *Theorems*.

But this *Problem* is better solved by the following :

THEOREM XVII.

As four Times the Area of any right-angled Triangle : the Hypotenuse :: the Hypotenuse : the Coscant of an Arc, which is always double to the Lesser of the acute Angles.

For Example ; suppose the Hypotenuse = 17 and the Area = 60, what are the other Parts of the Triangle ?

Say as $(4A =) 240 : \text{to } 17 :: 17 : 1,20416\text{c}$. (= the natural Coscant of an Arc, which is double to the lesser acute Angle) which answers in the Tables of Natural Co-Secants to $56^\circ . 8' . 55''$. whose half is $28^\circ . 4' . 27''$. Then as Radius : 17 :: Sine $28^\circ . 4' . 27''$: 8 the Lesser Leg of the Triangle, and therefore the greater Leg is 15. Q. E. I.

EXAMPLE VI.

Shewing the Use of *Theorem* 12. p. 13. in the Solution of Question 224. *Ladies Diary*, 1741. say as four Times 796 : $50 + 10 :: 50 - 10 : 575376\text{c}$. (= the Cotangent of half the Angle opposite to the Base) which answers in Natural Tables of Cotangents to $52^\circ . 59' . 34''$. whose Double is $105^\circ . 59' . 08''$. Now *per Theorem* 10. p. 13. as $50 : 10 :: \text{Sine } 37^\circ . 00' . 26'' : \text{Sine } 6^\circ . 54' . 51''$

D

the

the Difference of the Angles at the *Base*. Hence the lesser Angle at the *Base* is $30^{\circ} . 05' . 35''$. And now the *Problem* becomes the same as *Example 6. p. 25. in Simpson's Fluxions.* Q. E. D.

EXAMPLE VII.

Shewing the Use of *Theorem 13. p. 14.* in solving the 6th *Case of Plain Trigonometry*. Let the Sides of the Triangle be 562, and 320, and the Base 800 (see p. 14. *SHERWIN'S Mathematical Tables* applied to the Solution of the Cases in *Plain Trigonometry*). Then as $2 \times 562 \times 320 : 562 + 320 - 800$
 $\times 562 + 320 + 800 :: 1 : \text{versed Sine of the Sum}$
 of the Angles at the *Base*; that is, as $2 \times 562 \times$
 $320 : 82 \times 1682 :: 1 : \frac{82 \times 1682}{2 \times 562 \times 320} = \frac{41 \times 841}{562 \times 160} =$
 $= .38346\bar{6}c.$ which answers in the Natural Tables of versed Sines to $51^{\circ} . 56'$. which taken from 180° leaves $128^{\circ} . 04'$. the Angle opposite to the Base, viz. the Angle *ABC*. Q. E. I.

Note, The Rule for solving this Case by *Logarithms* is thus :

Add the three Sides together, and take half the Sum thereof ; this done, add the *Logarithms* of the half Sum, Difference betwixt the same half Sum and the Base ; and Radius, all into one Sum, and add the *Logarithms* of one of the Sides, to the *Logarithm* of half the other ; then if this Sum be taken from the former, the Remainder will be the versed Sine of the Sum of the Angles at the Base, which taken from 180° leaves the Angle sought.

For

For Example, in the Triangle *ABC* mentioned above,

AC the Base = 800

AB the Side = 562

BC the Side = 320

The Sum 1682

The half Sum 841 Log. 2.9247960

The Difference of *AC* 481 Log. 1.6127839

The Sum + Radius is = 14.5375799 = *A*

The Side *AB* = 562 Log. 2.7497363

Half the Side *BC* = 160 Log. 2.2041200

The Sum 4.9538563

which taken from *A*, leaves 9.5837236

which answers in the Table of *Logarithm* versed

Sines to $51^{\circ} . 46'$. And 180° Minus $51^{\circ} . 56' =$

$128^{\circ} . 04' =$ the Angle *ABC*. Q. E. I.

EXAMPLE VIII.

Shewing the Use of *Theorem* 14. p. 14. in the Solution of *Problem* the 4th in *NEWTON'S Algebra*, p. 103. supposing $BC + CA + AB = 440$. the Area of the Triangle $BAC = 6600$, and the Angle $BAC = 112^{\circ} . 37'$. Then per *Theorem* 14. as $440 : 13200 ::$ Co-Tangent $56^{\circ} . 18' . 30'' : 20,00001$, which taken from half 440, viz. 220 Leaves 200 = *BC* the required Base, which being known, the other Parts of the Triangle are easily found per foregoing *Theorems*. Q. E. D.

Also by this *Theorem* may easily be solved *Problem* 9. *WARD'S Mathematicks*, p. 320. For there the Cotangent of half the Angle opposite to the Hypothenufe being = 1, it will be as $234 : 4680 :: 1 :$

20, which being taken from half 234, leaves 97 for the required Hypothenufe. Q. E. I.

EXAMPLE IX.

Shewing the Use of *Theorem 15.* p. 14. in the
4 Solution of *Problem 56.* in NEWTON'S *Algebra*.
Supposing $AB + BC + CA = 600$, and $CD = 120$. Then because the Cosine of half the given Angle ACB is $= .7071068$ we have as $600 : 240 :: .7071068 : .2828427$ which added to $.7071068$ ($=$ Cosine of half the Angle opposite to the Base) makes $.989949$ which in the Tables of Natural Cosines gives $8^\circ . 08'$. which added to 45° gives $53^\circ . 08'$. $=$ Angle BAC , and taken from 45° , leaves $37^\circ . 08' =$ the Angle ABC . Now in the Triangles ADC and BDC is given the common Side CD , and all the Angles, to find the Sides AC , CB , DB and AD . Q. E. D.

Note, If the given Angle ACB had been any other Number of Degrees than 90 , the Solution would have been as easy.

EXAMPLE X.

Shewing the Use of *Theorem 16.* p. 14. in the Solution of the *Problem* proposed by Mr. MASON, *Ladies Diary 1731.* supposing the given Base $= 187$. the Perpendicular $= 204$, and the Angle opposite to the Base $= 30^\circ . 31'$. (Because if the given Angle be only 30° , as in the *Diary*, then the Question is easy by Mr. BULMAN'S Method). Then the Sum of the Angles at the Base is $149^\circ . 29'$. And per *Theorem* as $187 : 408 :: \text{Sine } 30^\circ . 31' : 1,107903\bar{6}c$. which taken from the versed Sine of $149^\circ . 29' = 1,8614815$ leaves $.753578\bar{6}c$. which in the Tables of natural versed Sines agrees to $75^\circ . 44'$. the Difference of the Angles at the Base,

Base, and so the Angles themselves become known; and then all the other Parts of the Triangle are found by the Rule of Three. Q. E. D.

Thus having given *Examples*, shewing the Use of the foregoing *Theorems*, we proceed to the following *Problems*, which cannot be solved by simple Equations (at least not by any Method I've yet discovered).

P R O B. VIII.

Having given the Base $EF = B$ (see Fig: p. 5.) of any plain Triangle, and the Sum of the Perpendicular and Sides, viz. $GE + GH + GF = n$; to find the Triangle; see Newton's Algebra, p. 163.

SOLUTION.

BECAUSE we have proved $qS = By$ and $2psy = Sy^2 - q^2S$ (see p. 6. l. 1. and 4th) and per Question $S + p = n$; therefore per first Equation find $S = \frac{By}{q}$, which substitute in the 2d and 3d, and we get $2psy = \frac{By^3}{q} - \frac{Bq^2y}{q}$ and $\frac{By}{q} + p = n \therefore p = \frac{qn - By}{q}$, which substituted for p in the foregoing Equation gives $2qns - 2sBy = By^2 - Bq^2 \therefore y^2 + 2sy = q^2 + \frac{2qsn}{B}$; adding s^2 to both Sides, makes $y^2 + 2sy + s^2 = s^2 + q^2 + \frac{2qsn}{B} = 1 + \frac{2qsn}{B}$ whose square Root is $y + s = \sqrt{1 + \frac{2qsn}{B}} \therefore y = \sqrt{1 + \frac{2qsn}{B}} - s$ = the Cosine of half the Difference of the Angles at

at the Base, and $p = \frac{qn - By}{q}$ becomes $p = n + \frac{s}{q} B$
 $-\frac{\sqrt{B^2 + 2qsnB}}{q}$ the Perpendicular sought; but
 if the given Angle $= 90^\circ$ then will $s = q = \sqrt{\frac{1}{2}B}$
 and then $p = n + B - \sqrt{2B^2 + 2nB} = GH$, and y
 $= \sqrt{1 + \frac{n}{B}} - s$. Q. E. I.

P R O B. IX.

*Given in any plain Triangle (see Fig. p. 5.)
 the Sum of the Sides $EG + GF = S$, and the
 Perpendicular $GH = p$, to find the Triangle;
 see Newton's Algebra, p. 164.*

SOLUTION.

WE have in the Solution of the foregoing Pro-
 blem $Sy^2 - q^2S = 2psy \therefore y^2 - \frac{2ps}{S}y = q^2$ and
 by adding $\frac{p^2s^2}{S^2}$ to both Sides it makes $y^2 - \frac{2ps}{S}y +$
 $\frac{p^2s^2}{S^2} = q^2 + \frac{p^2s^2}{S^2}$ whose Square Root is $y - \frac{ps}{S} =$
 $\sqrt{q^2 + \frac{p^2s^2}{S^2}} \therefore y = \frac{ps + \sqrt{q^2S^2 + s^2p^2}}{S}$ which may be
 better express'd; for s^2 is $= \frac{1}{2}$ the versed Sine of the
 Sum of the Angles at the Base, which call $\frac{1}{2}v$,
 then will $1 - s^2 = q^2 = 1 - \frac{1}{2}v$, which Values being wrote
 for s^2 and q^2 and we have $y = \frac{ps + \sqrt{S^2 - \frac{1}{2}v \times S^2 - p^2}}{S}$
 the Cosine of half the Difference of the Angles at
 the Base: Which will admit of a shorter Expression
 when

when the given Angle is $= 90^\circ$, for then $y =$
 $\frac{p \sqrt{2} + \sqrt{2s^2 + 2p^2}}{2s}$. And $\frac{s}{p + \sqrt{s^2 + p^2}} = EF$ the

Base of the required Triangle (which we presume is
 a much better Expression than that given by Sir

Isaac Newton, p. 165. l. 2. of his *Algebra*) or uni-
 versally its $\frac{2qs^2}{2sp + \sqrt{2cs^2 + 2op^2}}$ = the Base where

v = the versed Sine of the Sum of the Angle at the
 Base, and c = the versed Cosine of their Sum.
 Q. E. I.

P R O B. X.

In any plain Triangle, having given the Sum of
 Sides $= S$, and the Sum of the Perpendicular
 and Base $= m$, to find the Triangle. See
 Newton's *Algebra*, p. 165.

SOLUTION.

FROM p. 21. l. 15. where we have $qS = By$ and
 $2psy = Sy^2 - q^2S$ find $\frac{qS}{y} = B$ and $p = \frac{Sy^2 - S^2}{2sy}$

then will $B + p$ or $m = \frac{qS}{y} + \frac{Sy^2 - S^2}{2sy}$, which re-

duced is $y^2 - \frac{2sm}{S} = q^2 + 2sq$, solved $y =$

$\frac{sm + \sqrt{q^2S^2 + 2sqS^2 + s^2m^2}}{S}$ and $\frac{qS}{y} = B$ becomes

$\frac{qS^2}{sm + \sqrt{q^2S^2 + 2sqS^2 + s^2m^2}} = B$. But if the given

Angle be $= 90^\circ$, as in Sir Isaac Newton, then the
 Expression will be much neater; for then $s = q$.

And

And $\frac{S^2}{m + \sqrt{3S^2 + m^2}} = B$, the Base of the required Triangle. Q. E. I.

P R O B. XI.

Given $Ayx = By^2 + Dx^2$, to find x or y the Sine and Cosine of any required Arc or Angle.

SOLUTION.

THE Equation divided by yx makes $A = B \frac{y}{x} + D \frac{x}{y}$. But $\frac{x}{y} = t$ the Tangent of the required Arc, and $x = ty \therefore \frac{1}{t} = \frac{y}{x}$, whence by Substitution it is $A = \frac{B}{t} + Dt \therefore At = B + Dtt$, and so the Problem may now be solved by a Quad. Equation. Q. E. I.

P R O B. XII.

Given $Ay^2 + Byx + Dx^2 = E$, to find x the Sine, and y the Cosine of the requir'd Arc or Angle.

SOLUTION.

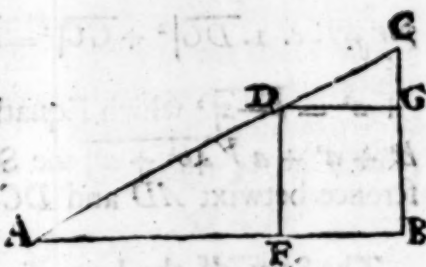
MULTIPLY E by $x^2 + y^2 = 1$, then the Equation becomes $Ay^2 + Byx + Dx^2 = Ex^2 + Ey^2$, which divided by yx , gives $A \frac{y}{x} + B + D \frac{x}{y} = E \frac{x}{y} + E \frac{y}{x}$, put $\frac{x}{y} = t$ the Tangent of the required

required Arc, then will $x = ty$ and $\frac{x}{y} = \frac{y}{x}$, and therefore by Substitution $A + Bt + Dtt = Ett + E$, and thus the Problem is solved by a quadratick Equation. Q. E. I.

P R O B. XIII.

By Mr. THOMAS BIRD.

In the rectangled Triangle ABC is given, the Hypotenuse $AC = 8,75 = b$, and the Side of the inscribed Square $DG = 6 = a$, to find the Sides of the Triangle.



SOLUTION.

H A L F the Sum of the Angles A and C is 45° whose Sine and Cosine call s , and the Sine and Cosine of half their Difference call x and y ; then per Theorem 1 and 2. p. 3. find $sy + sx =$ Sine of the Angle C the greater, and $sy - sx$ the Sine of the Angle A the Lesser: Then say, as

$sy + sx : a (DF) :: 1 (\text{Radius}) : CD = \frac{a}{sy + sx}$ and by a

like Way of Reasoning get $\frac{a}{sy - sx} = AD$. Hence

$$\frac{a}{sy - sx} + \frac{a}{sy + sx} = b = AC, \text{ out of Fractions it's}$$

$$ysa + sxa + sya - sxa = bs^2y^2 - bs^2x^2 \text{ contracted, } 2ay$$

E $= bsy^2$

$=bsy^2 - bsx^2$, and substituting $1 - y^2$ for x^2 , and bringing the y 's to one Side of the Equation, it's

$$2bsy^2 - 2ay = bs \therefore y^2 - \frac{a}{bs} y = \frac{1}{2} \text{ the Square completed, \&c. gives } = y \frac{a + \sqrt{a^2 + b^2}}{b\sqrt{2}} \text{ the Co-}$$

sine of half the Difference betwixt the Angles A and C . Q. E. I.

Otherwise put $2b = AC$, $2x = AD - DC$, then will $b + x = AD$, and $b - x = DC$. And per simi-

lar Triangles, $AD : FD :: DC : GC = \frac{ab - ax}{b + x}$ and

$$\text{per 47. e. i. } \overline{DG}^2 + \overline{GC}^2 = \overline{DC}^2 \text{ that is } \frac{a^2 x^2 - a^2}{b + x}^2$$

$+ a^2 = \overline{b - x}^2$ which Equation, ordered gives $x^2 = b^2 + a^2 + a \sqrt{4b^2 + a^2}$ the Square of half the Difference betwixt AD and DC . Q. E. I.

The Sum of the Legs viz. $AB + BC$ may be had by this Theorem $S = a + \sqrt{a^2 + b^2}$. (See p. 6. l. 1.)

And then their Difference by this, $D = \sqrt{2b^2 - S^2}$. Other Methods of Solution from our ingenious Correspondents will be given further on.

N. B. Since the foregoing Problems were sent to the Press, I have thought upon better Solutions to the following than I could make when I finished those on simple Equations, which I hope will be acceptable to the Reader in this Place.

P R O B.

P R O B. XIV.

In any plain Triangle, as EGF, (See Fig. p. 5.) there is given the Difference of the Sides EG—GF=D, the Difference of the Segments of the Base EH—HF=b, and the Angle EGF, to find the other Parts of the Triangle?

S O L U T I O N.

PUT s and q to express the Sine and Cosine of half the Sum of the Angles at the Base, and x and y for those of half their Difference, and p for the Perpendicular GH. Then per Trigonometry, as Sine Angle HEG : GH :: Sine Angle HGE : EH;

that is, $sy - qx : p :: qy + sx : \frac{pqy + psx}{sy - qx} = EH.$

And Sine Angle HFG : HG :: Sine Angle HGF : HF; that is, $sy + qx : p :: qy - sx : \frac{pqy - psx}{sy + qx} =$

HF. Hence $\frac{pqy + psx}{sy - qx}$ minus $\frac{pqy - psx}{sy + qx} = EH -$

HF = b , which reduced to a common Denominator

and Terms contracted, becomes $\frac{2pyx}{s^2 - x^2} = b \therefore \frac{2pyx}{b} =$

$s^2 - x^2$. But we have proved $2sqp = Bs^2 - Bx^2$

(See p. 7. l. 1.) $\therefore \frac{2sqp}{B} = s^2 - x^2$; consequently

$\frac{2pyx}{b} = \frac{2sqp}{B} \therefore 2yxB = 2sqb$. But it is proved at

p. 6. l. 20. $xB = sD$; whence by substituting sD for xB we get $2ysD = 2sqb \therefore yD = qb$, or as $D : b ::$

E 2

q : y,

$q : y$, the Cosine of half the Difference of the Angles at the Base.

Or in Words it is THEOREM 18.

As the Difference of the Sides of any plain Triangle : the Difference of the Segments of the Base :: the Cosine of half the Sum of the Angles at the Base : the Cosine of half their Difference.

P R O B. XV.

In any plain Triangle, (See Fig. p. 5.) let there be given the Base $EF=B$, the Difference of the Segments of the Base, viz. $EH-HF=b$, and the Angle EGF opposite to the Base, to find the other Parts of the Triangle?

S O L U T I O N.

LET the (known and unknown) Quantities express the same as in the Solution to the foregoing Problem, then will $2y \times B = 2sqb$; (See p. 27. l. 21.) or as $B : b :: 2sq : 2y \times$ the Sine of the Difference of the Angles at the Base.

Which put into Words, makes THEOREM 19.

As the Base of any plain Triangle : the Difference of the Segments of the Base :: the Sine of the Sum of the Angles at the Base : the Sine of their Difference.

P R O B.

P R O B. XVI.

In the Triangle EGF. (See Fig. p. 5.) is given the Difference of the Sides $EG - GF = D$, the Difference of the Segments of the Base $EH - HF = b$, and the Perpendicular $GH = p$, to find the other Parts of the Triangle? (See Ward's Mathematicks, p. 322.)

SOLUTION.

BECAUSE we have proved $\frac{2pyx}{b} = s^2 - x^2$, or which is the same $\frac{2pyx}{b} = y^2 - q^2$; and also $yD = qb$. (See l. 18. and last, of the 27th Page) therefore by writing $\frac{y^2 D^2}{b^2}$ for q^2 we get $\frac{2pyx}{b} = y - \frac{y^2 D^2}{b^2} \therefore 2pxb = b^2 y - yD^2 \therefore \frac{x}{y} = \frac{b^2 - D^2}{2pb}$, or as $2pb : b + D :: b - D : \frac{x}{y}$ the Tangent of half the Difference of the Angles at the Base. And then $b : D :: y : q$ the Cosine of half the Sum of the Angles at the Base.

Which in Words is THEOREM 20.

1. As the double Rectangle of the Perpendicular, and Difference of the Segments of the Base : the Difference of the Sides, *Plus* Difference of the Segments of the Base :: the Difference of the Segments of the Base, *Minus* the Difference of the Sides : the Tangent of half the Difference of the Angles at the Base.

2. Then as the Difference of the Segments of the Base : the Difference of the Sides :: the Cosine of half

half the Difference of the Angles at the Base : the
Cosine of half their Sum.

EXAMPLE II. See Ward, p. 322.

Where we have $D = 405$. $b = 495$. and $p = 300$. And per foregoing Theorem $2pb : b + D :: b - D : \frac{x}{y}$; that is, $297000 : 900 :: 90 : \text{Tangent of } 15^\circ . 15' . 18''$. = half the Difference of the Angles at the Base.

And per latter Part of the Theorem $b : D :: y : q$; that is, $495 : 405 :: \text{Cosine } 15^\circ . 15' . 18'' : \text{Cosine } 37^\circ . 52' . 29''$ = half the Sum of the Angles at the Base. Hence the Angles themselves are $22^\circ . 37' . 11''$. and $53^\circ . 07' . 48''$. And per Trigonometry, the Sides of the Triangle are found 375 and 780. And the Base 720. Q. E. I.

P R O B. XVII.

The Sum of two Sides of any plain Triangle, the Difference of the Segments of the Base, and the Angle opposite to the Base being given, to find the other Parts of the Triangle?

SOLUTION.

FOR the Sum of the two Sides put S , and let all the other Letters express the same as in the last Problem ; then because $\frac{2pyx}{b} = y^2 - q^2$. (See p. 27. l. 18.) and $2psy = Sy^2 - q^2S$, (See p. 6. l. 4.) or $\frac{2psy}{S} = y^2 - q^2$; consequently $\frac{2psy}{S} = \frac{2pyx}{b} \therefore sb = Sx$,
or

or as $S : b :: s : x$ the Sine of half the Difference of the Angles at the Base. (Which being known we may *per* Equation, p. 6. l. 1. say as $y : q :: S : B$ the Base of the required Triangle).

Or in Words it is THEOREM 21.

As the Sum of the Sides of any plain Triangle : the Difference of the Segments of the Base : : the Sine of half the Sum of the Angles at the Base : the Sine of half their Difference.

P R O B. XVIII.

The Sum of the two Sides of any plain Triangle, the Difference of the Segments of the Base, and the Perpendicular let fall from the vertical Angle upon the Base being given, thence to find the rest of the Triangle?

SOLUTION.

SINCE $\frac{2pyx}{b} = s^2 - x^2$, and $sb = Sx$. (See p. 30. l. 24.

and last) we have by substituting $\frac{S^2x^2}{b^2}$ for s^2 , $\frac{2pyx}{b} =$

$\frac{S^2x^2}{b^2} - x^2 \therefore 2pby = S^2x - b^2x \therefore \frac{y}{x} = \frac{S^2 - b^2}{2pb}$. Or as

$2pb : S + b :: S - b : \frac{y}{x}$, the Cotangent of half the

Difference of the Angles at the Base. And $b : S :: x : s$, the Sine of half the Sum of the Angles at the Base. Q. E. I.

From

From $\frac{y}{x} = \frac{S^2 - b^2}{2pb}$ we get THEOREM 22.

In any plain Triangle, as the double Rectangle of the Perpendicular, and Difference of the Segments of the Base: the Sum of the Sides, *Plus* the Difference of the Segments of the Base :: the Sum of the Sides, *Minus* the Difference of the said Segments: the Cotangent of half the Difference of the Angles at the Base.

EXAMPLE. XI.

Shewing the Use of the foregoing Theorem in the Solution of Problem 13. *Ward's Mathematicks*, p. 322. where $S = 1155$. $b = 495$. and $p = 300$. Whence *per Theorem 18.* as $297000 : 1650 :: 660 : 3,666\bar{6}$. the Cotangent of half the Difference of the Angles at the Base. Which answers in the Tables to $15^\circ . 15' . 18''$. And $495 : 1155 :: \text{Sine } 15^\circ . 15' . 18'' : \text{Sine } 37^\circ . 52' . 29''$. (See p. 31. l. 21.) = half the Sum of the Angles at the Base; which being known, the Angles themselves are found $22^\circ . 37' . 11''$. and $53^\circ . 07' . 47''$. and as $\text{Sine } 22^\circ . 37' . 11'' : 300 :: \text{Radius} : 780$ the greater Side. Hence the lesser Side is 375. Q. E. I.

P R O B. XIX. By S. D.

Suppose the Angle $EGF = 90^\circ$. (See Fig. p. 5.) and there be given the Difference of the Legs $EG - GF = d$, and the Ratio of the Segment $EH : HF$, as $a : b$, to find the Triangle?

SOLUTION.

LET x and y = the Sine and Cosine of the Angle HEG , then will y and x = Sine and Cosine of the Angle HFG , and let $p = GH$; then
per

per Trigonometry $x : p :: y : \frac{py}{x} = EH$, and $y : p ::$

$x : \frac{px}{y} = HF$; but as $\frac{py}{x} : \frac{px}{y} :: a : b \therefore \frac{bpy}{x} = \frac{pxa}{y}$

$\therefore by^2 = ax^2 \therefore \frac{b}{a} = \frac{x^2}{y^2}$, the Square of the Tan-

gent of the Angle HEG . Again, if z be put for FG , then will $z + d = EG$; and per Trigonome-

try $x : z :: y : z + d \therefore yz = zx + xd \therefore z = \frac{xd}{y-x}$

$= GF$. Q. E. I.

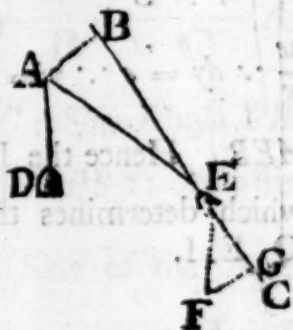
PROB. XX.

By the late Sir, ISAAC NEWTON.

If at the Ends of the Thread DAE , moving upon the fix'd Tack A , there are banded two Weights D and E , whereof the Weight E slides through the oblique Line BG given in Position; to find the Place of the Weight E , where these Weights are in Equilibrio.

SOLUTION.

SUPPOSE the Problem done, and parallel to AD draw EF , which shall be to AE as the Weight E to the Weight D . And from the Points A and F to the Line BG let fall the Perpendiculars AB , FG . Now since the Weights are, by Supposition, as the Lines AE and EF , exprefs those Weights by those



those Lines, the Weight D by the Line EA , and the Weight E by the Line EF . Therefore the Body, or E , directed by the Force of its own Weight, tends towards F , and by the oblique Force EG tends towards G . And the same Body E by the Weight D , in the direct Force AE , is drawn towards A , and in the oblique Force BE is drawn towards B . Since therefore the Weights sustain each other in Equilibrio, the Force by which the Weight E is drawn towards B , ought to be equal to the contrary Force by which it tends towards G ; that is, BE ought to be equal to EG . But now the Ratio of AE to EF is given by the Hypothesis, and by Reason of the given Angle FEG , there is also given the Ratio of FE to EG , to which BE is equal. Therefore there is given the Ratio of AE to BE . AB is also given in Length; and thence the Triangle ABE , and the Point E will easily be given, viz. make $AB = a$, and put x and y to express the Sine and Cosine of the Angle AEB , Radius = 1. then $x : a :: 1 : AE = \frac{a}{x}$ and $x : a :: y$

$: BE = \frac{ay}{x}$; moreover let AE be to BE in the given

Ratio of d to e ; then it will be as $d : e :: \frac{a}{x} : \frac{ay}{x}$ and

multiplying Extreams and Means, we have $\frac{day}{x} =$

$\frac{ae}{x} \therefore dy = e \therefore y = \frac{e}{d}$ the Cosine of the Angle

AEB . Hence the Length $BE (= \frac{ay}{x})$ is found,

which determines the Place of the Weight E .
Q. E. I.

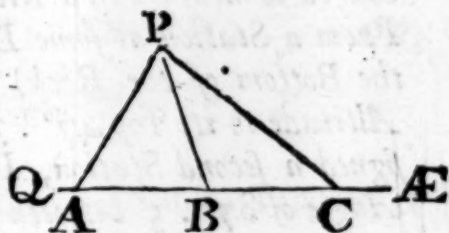
P R O B. XXI.

By Mr. ROBERT HEATH.

Three Ships *A, B, C*, sail from a certain Place in North Latitude, until they arrived at three different Ports, all lying under the Equinoctial, *A* sail'd a direct Course between the South and the West, 175,62 Leagues, *C* sail'd 133 Leagues between the South and the East, *B* sail'd a Course between *A* and *C*, 102 Leagues, making the Angle or Rhumb with *A* = the Angle *C* made with the Equinoctial. Hence it is required to find the Port sail'd from, each Ship's Course, Distance from each other, and their respective Ports.

SOLUTION.

LET *QÆ* represent the Equator, put $a = AP$, $b = BP$, $c = CP$, $e = AB$, and $x =$ Sine of the Angle *APB* or *BCP*. Then per Trigonometry as



$BP : \text{Sine Angle } BCP :: PC : \text{Sine Angle } PBC = \frac{cx}{b}$, and as $AB : \text{Sine Angle } APB :: AP : \text{Sine}$

Angle $ABP = \frac{ax}{e}$. But the Sine of the Angle

ABP is = Sine of Angle PBC . Hence $\frac{cx}{b} = \frac{ax}{e}$

$$\therefore \frac{c}{b} = \frac{a}{e} \therefore e = \frac{ba}{c}, \text{ or as } PC : PB :: PA : AB.$$

And now there will be given all the Sides to find *per Theorem 13. p. 14.* the versed Sine of the Sum of the Angles at the Base, *viz.* the Angles *A* and *B*, which taken from 180° , leaves the Angle *APB*. Hence the

$$\text{Course of } \left\{ \begin{array}{l} A \ 54^\circ . 38' \text{ West} \\ B \ 4 \ . 47 \text{ West} \\ C \ 40 \ . 10 \text{ East} \end{array} \right\} \begin{array}{l} \text{or} \\ \text{Diff.} \\ \text{Lon.} \end{array} \left\{ \begin{array}{l} 7 \ . 9 \\ 0 \ . 25 \\ 4 \ . 7 \end{array} \right\} \text{Req.}$$

P R O B. XXII.

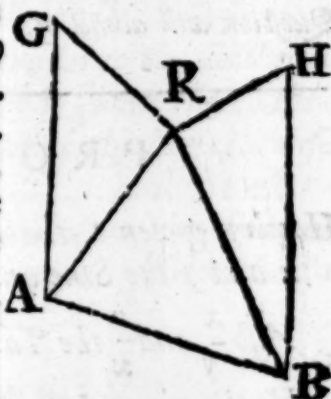
By Mr. HENRY TRAVIS.

At Matlock, near the Peak in Derbyshire, where are many surprizing Curiosities in Nature, is a Rock by the Side of the River Derwent, rising perpendicularly to a wonderful Height; which, being inaccessible, I endeavoured to measure in a Mathematical Method. From a Station at some Distance, (level with the Bottom of the Rock) I took an Angle of Altitude to its Top $47^\circ . 30'$; and having designed a second Station, I took an Horizontal Angle of $87^\circ . 5'$ between the Foot of the Rock and that Station; the measured Distance between the Stations was 4 Chains, and 29 Links, (per Gunter) or 283,274 Feet. At that Place I had an Angle of Altitude $40 \ . 12'$, but forgot from hence to take an Angle between my first Station and the Foot of the Rock; yet am in Hopes some curious Artist will, from this
Data,

Data, determine the perpendicular Height of this stupendious Rock, with a general Theorem for all Questions of this Nature?

SOLUTION.

IN the annexed Figure is given $AB = b = 4,29$ the Angle $BAR = 87^\circ . 5'$. the Angle $GAR = 47^\circ . 30'$. the Angle $HBR = 40^\circ . 12'$. the Angle $BRH = 90^\circ$ = the Angle ARG , and the Line $GR = RH$.



Put t = the Tangent of the Angle GAR , and T = that of HBR , m = the Sine Angle RAB , and y = that of ABR , and let $x = RB$; then per Trigonometry, as Sine $BAR : RB ::$ Sine

$ABR : AR = \frac{yx}{m}$, and as Radius (r) : Tangent

Angle $RAG :: AR : GR = \frac{yxt}{rm}$. Again Radius (r)

: Tangent Angle $RBH :: RB : RH = \frac{Tx}{r} = GR$,

per Question, consequently $\frac{yxt}{rm} = \frac{Tx}{r} \therefore \frac{yt}{m} = T \therefore$

$y = \frac{Tm}{t}$, or as $t : T :: m : y$, that is, Tangent of

Angle GAR : Tangent Angle $HBR ::$ Sine Angle RAB : Sine of the Angle ABR , which being known we find the Angle $ARB =$. Then Sine of Angle $ARB : AB ::$ Sine Angle $RAB : RB (=x)$. And Lastly, Radius : Tangent Angle $RBH :: RB : RH$. Q. E. I.

From the Solutions of the four last Problems the Reader may observe, that it is often the best Way to make

make use of two (or more) unknown Quantities in the Solution of a Problem; when the Solution may easily be had with one unknown Quantity: Because there is no universal Rule can be given for the Election of Quantities which we are to make use of for the Calculus; so as to get the most elegant Solution that the Question will admit of.

P R O B. XXIII.

Having given t , the Tangent of an Arc, to find x and y the Sine and Cosine of double that Arc.

Also $\frac{x}{y}$ and $\frac{y}{x}$ the Tangent and Cotangent of double the given Arc, &c.

SOLUTION.

PUT $n = \sqrt{1+t^2}$ to express the Secant of the given Arc, then will $\frac{1}{n}$ = the Cosine of the said Arc, and $\frac{t}{n}$ = its Sine (Radius being 1) and per Theorem 1. and 2. p. 3. we get $\frac{2t}{n^2}$ or $\frac{2t}{1+t^2} = x$ and $\frac{1-t^2}{n^2}$ or $\frac{1-t^2}{1+t^2} = y$ the Sine and Cosine of the required Arc. Hence $\frac{x}{y} = \frac{2t}{1-t^2}$ and $\frac{y}{x} = \frac{1-t^2}{2t}$, the Tangent and Cotangent of the required Arc. And Lastly, $\frac{1}{y} = \frac{1+t^2}{1-t^2}$ and $\frac{1}{x} = \frac{1+t^2}{2t}$ is the Secant and Cosecant of the required Arc. Q. E. I.

P R O B.

P R O B. XXIV.

Being the same with Prize Question, Diary 1725.

There is a right-angled Triangle, where in a right Line is drawn parallel to the Cathetus (or Perpendicular) there is given the Cathetus $= c$ (24,) that Segment of the Hypotenuse next to the Cathetus $= b$ (15,) and the alternate Segment of the Base $= b$ (20,) to find the other Parts of the Triangle. (See Ward's Mathematicks, p. 328.)

SOLUTION.

FOR the Sine and Cosine of the Angle opposite to the given Cathetus put x and y . (See the Fig. in Ward) Radius $= 1$, then $y : b : x :$

$\frac{bx}{y} =$ Part of the Cathetus, and $1 : b :: x : bx =$

the other Part. Hence $\frac{bx}{y} + bx = c \therefore bx + bxy$

$= cy$, or by Writing $\frac{2t}{1+t^2}$ for x , and $\frac{1-t^2}{1+t^2}$ for y ,

it's $\frac{2bt}{1+t^2} + \frac{2bt - 2bt^3}{1+t^2} = \frac{c - ct^2}{1+t^2}$ which re-

duced gives $\frac{+2b}{+2b} \left\{ \begin{matrix} t \\ t^3 \end{matrix} \right\} + \frac{+2b}{-2b} \left\{ \begin{matrix} t \\ t^3 \end{matrix} \right\} + ct^2 = c$. In Num-

bers it is $35t + 5t^3 + 12t^4 = 12$, which Equation solved gives $t = .333447\&c.$ which answers in the Tables of Natural Tangents to $18^\circ . 26' . 27''$. which doubled is $36^\circ . 52' . 54'' =$ the Angle opposite to the Cathetus. Q. E. I.

* See p. 38. l. 18.

P R O B.

P R O B. XXV.

By Mr. W. DEARE, Diary 1720.

In any plain Triangle, (see Fig. p 5.) there is given the Side $GF = a$, and the Segment $EH = b$, and the Angle $EGF =$ any Number of Degrees, to find the other Parts of the Triangle.

SOLUTION.

LET x and y express the Sine and Cosine of the Angle HGE , and s and q those of the Angle EGF , (Radius = 1) then per Theorem 4. p. 3. $qy + sx =$ Cosine of the Difference betwixt the Angle EGF and EGH , viz. the Cosine of HGF , or Sine of HFG ; then per Trigonometry, as $x : b :: y : \frac{by}{x} = GH$, and $1 : a :: qy + sx : aqy + asx = GH$,

consequently $\frac{by}{x} = aqy + asx$, and by substituting

$$\frac{2t}{1+t^2} \text{ for } x; \frac{1-t^2}{1+t^2} \text{ for } y, \text{ and } \frac{1-t^2}{2t} \text{ for } \frac{y}{x}, \text{ we get}$$

$$\frac{b-bt^2}{2t} = \frac{aq-aqt^2}{1+t^2} + \frac{2ast}{1+t^2}, \text{ which Equation re-}$$

duced, gives $2aqt + 4ast^2 - 2aqt^3 + bt^4 = b$, and solv'd gives the Value of t , the Tangent of half the Angle HGE . Q. E. I.

* See p. 38. l. 19, and 20.

P R O B. XXVI.

In any right-angled Triangle, the Area, and Sum of the Hypotenuse, when added to either Side, being given, thence to find the Sides, &c. (See Ward, p. 326.)

S O L U T I O N.

SUPPOSE the Area = 1350 = A , 120 = s the Sum, &c. And put x and y = the Sine and Cosine of the lesser acute Angle, Radius = 1. then per 19. e. 6, as $s^2 : 2A :: 1 + y^2 : yx$. Whence by multiplying Extreams and Means, we have $yx s^2 =$

$$2A x \sqrt{1 + y^2} \therefore \frac{yx}{1 + y^2} = \frac{2A}{s^2}. \text{ But } y = \frac{1 - t^2}{1 + t^2} \text{ and}$$

$$x = \frac{2t}{1 + t^2} \text{ (see p. 38. l. 18.) which being substituted for } x \text{ and } y, \text{ makes } \frac{1 - t^2}{2} = \frac{2A}{s^2} \text{ therefore } 1 - t^2$$

$$= \frac{4A}{s^2}, \text{ which Equation solved, gives } t = 33344 \text{ \&c.}$$

which answers in the Tables of natural Tangent to $18^\circ . 26' . 27''$. = half the lesser acute Angle.
Q. E. I.

G

P R O B.

P R O B. XXVII.

In any plain Triangle, as EGF, (see Fig. p. 5.) there is given the Sum of the Sides FE + EG = n, the Area of the Triangle = A, and the Angle EFG = any Number of Degrees whatsoever, to find the other Parts of the Triangle ?

SOLUTION.

PUT s and q for the Sine and Cosine of half the Sum of the Angles E and EGF , x and y for the Sine and Cosine of half their Difference; then will $sy - qx =$ Sine of the Angle E , and $sy + qx$ will be that of the Angle EGF . Then per 19. e. 6. $n^2 : A :: 2sq + sy + qx|^2 : qs^2 - sqx^2$, and by multiplying Extreams and Means it's $n^2qs^2 - n^2sqx^2 = A^2 2sq + sy + qx|^2$. But $\frac{2t}{1+t^2} = x$ and $\frac{1-t^2}{1+t^2} = y$, (see p. 38. l. 18.) which Values being put for x and y , and the Equation ordered, will produce a Biquadratic Equation, which being solved, will give the Value of t .

Otherwise, if x be put for the Base EF , s and c for the Sine and Cosine of the given Angle, Radius = 1, and the other Quantities to express as before; then will $\frac{2A}{x} =$ the Perpendicular GH . And per

Trigonometry $s : \frac{2A}{x} :: c : HF = \frac{2cA}{sx} \therefore x = \frac{2cA}{sx}$
 $= EH$. Again $s : \frac{2A}{x} :: 1 : GF = \frac{2A}{sx} \therefore n = \frac{2A}{sx}$
 $= GE$. But per 47. e. 1. $\overline{EH}^2 + \overline{HG}^2 = \overline{EG}^2$,
 that

that is $x^2 - \frac{4cA}{s} + \frac{4c^2A^2}{s^2x^2} + \frac{4A^2}{x^2} = n^2 - \frac{4nA}{sx} + \frac{4A^2}{s^2x^2}$

which Equation reduced gives $x^4 - 4\frac{A}{s}x^2 + \frac{4A^2}{s^2} = n^2x^2 - 4\frac{nA}{s}x + \frac{4A^2}{s^2}$
 $= \frac{4A^2 - c^2A^2}{s^2} - 4A^2$, which solved will give the
 Value of $x = EF$. Q. E. I.

Note, If the given Angle $EFG = 90^\circ$ as in Ward, p. 326. then is $s = 1$. and $c = 0$, consequently the Equation will then become (*) $x^3 - nx^2 + 4nA = 0$, the same Equation with that in Ward.

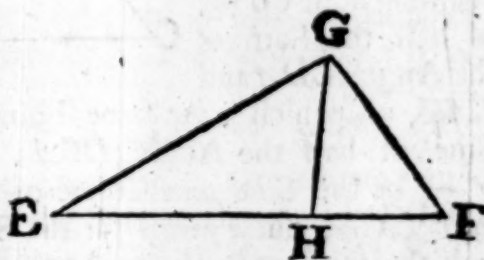
P R O B. XXVIII.

There is an oblique-angled plain Triangle, where-
 in a Perpendicular is let fall from the vertical
 Angle upon the Base; the Sum of each Seg-
 ment of the Base when added to its adjacent
 or next Side, and the Area of the Triangle
 are given, to find the Perpendicular, and each
 Side. (See Ward, p. 327.)

SOLUTION.

PUT x for
 the Perpen-
 dicular GH , $n =$
 $GE + EH$, $m =$
 $HF + FG$, $a =$
 the Area of the
 Triangle HEG ,

and $y =$ the Area of HFG ; then by the Equation
 resulting from the Solution of the last Problem (*)



G 2

we

we get $a = \frac{n^2x - x^3}{4n}$ and $y = \frac{s^2x - x^3}{4s}$. Hence

$(a + y \text{ is } =) \frac{n^2x - x^3}{4n} + \frac{s^2x - x^3}{4s} = A$ the Area of the Triangle EFG , $\therefore sn^2x - sx^3 + ns^2x - nx^3 = 4snA$, which Equation solved gives the Value of x .
Q. E. I.

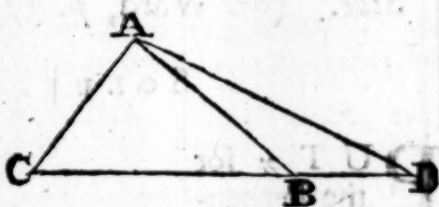
P R O B. XXIX.

Being the same with the Prize Question in Diary 1723.

In the oblique-angled Triangle CAD, there is given the Side AD=68, and the Sum of the Sides AC + CD=148; also within the Triangle there is given the Line AB perpendicular to the Side CA. Thence to find the other Parts of the Triangle?

SOLUTION.

PUT $CA + CD = S$, $AD = d$,
 $AB = b$; s and q for
the Sine and Cosine
of half the Sum of
the Angles CDA and



CAD , or, which is the same Thing, the Cosine and Sine of half the Angle DCA ; then will $2sq$ and $q^2 - s^2$ be the Sine and Cosine of the Angle DCA . And Lastly, put x and y for the Sine and Cosine of half the Difference of the Angles CDA and CAD ; then will $sy - qx$ be the Sine of the Angle D . Then Theorem 9. p. 13. as $d : S :: q : y \therefore dy = Sq$; again

as

as $2sq : d :: sy - qx : CA = \frac{dsy - dqx}{2sq}$ and as $2sq : b$

$:: q^2 - s^2 : CA = \frac{bq^2 - bs^2}{2sq}$, consequently $dsy - dqx$

$= bq^2 - bs^2$, or by writing Sq for dy , it's $Sqs - dqx =$

$bq^2 - bs^2 \therefore Sqs + bs^2 - bq^2 = dqx$, which squared makes

$S^2q^2s^2 + 2Sbqs^2 - 2Sbqsq^2 - 2b^2q^2s^2 + b^2s^4 + b^2q^4 = d^2qx^2$

$= d^2q^2 - S^2q^4$ (because $d^2y^2 = S^2q^2$, or since y^2 is $=$

$1 - xx$, $d^2 - d^2x^2 = S^2q^2 \therefore d^2 - S^2q^2 = d^2x^2$ &c.) which

Equation divided by q^2s^2 , makes $S^2 + 2Sb \frac{s}{q} - 2Sb \frac{q}{s}$

$- 2b^2 + b^2 \frac{s^2}{q^2} + b^2 \frac{q^2}{s^2} = d^2 \frac{1}{s^2} - S^2 \frac{q^2}{s^2}$. Put $t = \frac{s}{q}$ the

Tangent of half the Sum of the Angles D and

CAD , then will $\frac{1}{t} = \frac{q}{s}$ and $\frac{1+t^2}{t^2} = \frac{1}{s^2}$, which be-

ing substituted in the Equation above, gives

$S^2 + 2Sbt - 2Sb \frac{1}{t} - 2b^2 + b^2t^2 + b^2 \frac{1}{t^2} = \frac{d^2 + d^2t^2}{t^2}$

$- S^2 \frac{1}{t^2} = \frac{d^2 + d^2t^2}{t^2} + 2Sbt^3 - 2Sbt - 2b^2t + b^2t^4 + b^2 = d^2 +$

$d^2t^2 - S^2$. Hence $b^2t^4 + 2Sbt^3 - 2b^2 \left. \begin{matrix} + S^2 \\ - d^2 \end{matrix} \right\} t^2 - 2Sbt + b^2 \left. \begin{matrix} - d^2 \\ + S^2 \end{matrix} \right\}$

$= 0$, and this Equation solved, gives $t = ,600003 \&c.$

the Tangent of $30^\circ . 57' . 50'' =$ half the Sum of

the Angles CDA and CAD ; and *per Theorem 9. p.*

13. $y = \frac{Sq}{d}$ Cofine of $5^\circ . 55' . 04'' =$ half their

Difference. Hence the Angles are known, and

per Trigonometry, as the Sine of the Angle DCA :

$AD ::$ Sine of the Angle $CDA : CA = 48$. Q. E. I.

N. B.

N. B. If the Equation at 21st Step in p. 330 of *Ward's Mathematicks*, was reduced to a *General Theorem* as that above, it would stand thus :

$$\left. \begin{array}{l} -2S^2 \\ -4d^2 \\ +d^4 \end{array} \right\} a^4 + \left. \begin{array}{l} +4d^2 \\ -4S^3 \\ -8bS \end{array} \right\} a^3 + \left. \begin{array}{l} +8b^2S^2 \\ -4b^2d^2 \\ +S^4 \\ -2d^2S^2 \end{array} \right\} a^2 + \left. \begin{array}{l} +4b^2d^2S \\ -4b^2S^3 \\ +b^2S^4 \\ -2b^2d^2S^2 \end{array} \right\} a + \left. \begin{array}{l} +b^2d^4 \\ +b^2d^4 \end{array} \right\} = 0, \text{ where } a = AC, \text{ and the other Letters as in the foregoing Solution.}$$

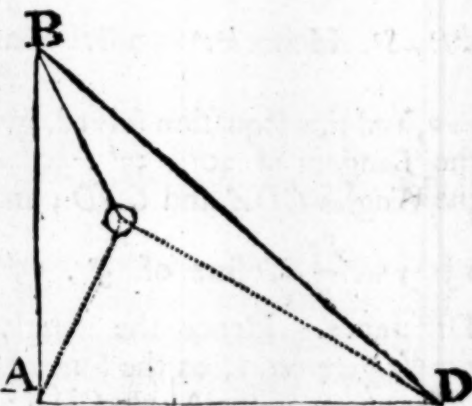
P R O B. XXX.

Being the Prize Question in Ladies Diary 1727.

In the Triangle *ABD* is given the two Sides *BA* = 80 = *b*, *BD* = 108 = *d*, with the included Angle *ABD* = 55 Degrees, as also *BO* = 50 = *p*; required to draw the Lines *AO* and *OD*, so that the Angle *AOB* may be equal to the Angle *DOB*?

SOLUTION.

PUT *s* and *q* to express the Sine and Cosine of half the given Angle *ABD*, which is the Sum of the Angles *ABO* and *OBD*; and let *x* and *y* be the Sine and Cosine of half their Difference, Radius = 1, then will *sy - qx* and *qy + sx* be the Sine and Cosine of *OBD*, (if the Angle *OBD* be less than the Angle *OBA*.) Then per Theorem,



p. 11. l. 16. find $\frac{dsy - dqx}{p - dqy - dsx} =$ the Tangent of the Angle DOB , and $\frac{bsy + bqx}{p - bqy + bsx} =$ that of AOB ,

which per Question are equal. Hence $\frac{bsy + bqx}{p - bqy + bsx} = \frac{dsy - dqx}{p - dqy - dsx}$, which multiplied out of Fractions, and the Equation contracted, makes $pdqx - dpsy = 2dbyx - pbsy - pbqx$. And by writing $\frac{2t}{1+t^2}$ for x , and $\frac{1-t^2}{1+t^2}$ for y ; (see page 38. line 18.) and reducing the Equation, we have

$$\left. \begin{array}{l} + 2pdq \\ + 2pbq \\ - 4db \end{array} \right\} t + \left. \begin{array}{l} + 2pdq \\ + 2pbq \\ + 4db \end{array} \right\} t^3 + \left. \begin{array}{l} pds \\ - pbs \end{array} \right\} t^4 = \left. \begin{array}{l} + pds \\ - pbs \end{array} \right\} \text{ which}$$

Equation being solved, will give the Value of t , which is the Tangent of one fourth Part of the Difference betwixt the Angles OBA and OBD .
Q. E. D.

If the Angle OBD be greater than the Angle OBA , then $sy - qx$ and $qy + sx$, which in Lines 3 and 4 are said to represent the Sine and Cosine of the Angle OBD , since they always belong to the greater of the two, will stand for the Sine and Cosine of the Angle OBA , whence it will come to pass, that in what Manner soever $sy - qx$ and $qy + sx$ were related to d in the preceding Solution, they will now be to b ; and however $sy + qx$ and $qy - sx$ were affected with b , they will now be with d ; and so by writing every where d for b , or b for d , and retaining the Signs in the foregoing final Equation, the Theorem will stand thus:

$$\left. \begin{array}{l} + 2pbq \\ + 2pdq \\ - 4bd \end{array} \right\} t + \left. \begin{array}{l} + 2pbq \\ + 2pdq \\ + 4dd \end{array} \right\} t^3 + \left. \begin{array}{l} pbs \\ - pds \end{array} \right\} t^4 = \left. \begin{array}{l} + pbs \\ - pds \end{array} \right\} \text{ where}$$

t is the same as before.

N. B.

N. B. *The Solution to the foregoing Problem is said to have baffled the ablest Mathematicians in Europe. See the Problem in RONAYNE's Algebra p.*

P R O B. XXXI.

By Mr. ROBERT FEARNSIDE,
being the Prize Question in the Ladies Diary 1733.

Given in the right angled Triangle ABE, the Base AB = 95,23, the Angle DAE = 15° the Line CA bisecting the Angle BAE, and making BC = CD; to find all the other Parts of the Triangle.

SOLUTION.

PUT $AB = b$, Sine of the Angle $EAD = s$, Sine of the Angle $CAB = EAC = x$; then the Sine of double the Angle CAB , i. e. EAB will be $2x\sqrt{1-xx}$, the Co-sine $1-2xx\sqrt{1-xx}$, $\therefore b ::$

$$x : CB = \frac{bx}{\sqrt{1-xx}} \text{ and } AC =$$

$$\frac{b}{\sqrt{1-xx}}, DB = \frac{2bx}{\sqrt{1-xx}},$$

$$DA = b \frac{\sqrt{1+3xx}}{1-xx}, \text{ then } 1-2xx : b \frac{\sqrt{1+3xx}}{1-xx} :: s$$

$$\therefore ED = \frac{bs}{1-2xx} \frac{\sqrt{1+3xx}}{1-xx}, \text{ then } CE \text{ will be } =$$

$$bx$$



$$\frac{bx}{\sqrt{1-xx}} + \frac{bs}{1-2xx} \frac{\sqrt{1+3xx}}{1-xx}; \text{ and } 1-2xx : b ::$$

$$1 : EA = \frac{b}{1-2xx} \text{ then per p. 3. 6 Eucl. } b : \frac{b}{1-2xx}$$

$$:: \frac{bx}{\sqrt{1-xx}} : \frac{bx}{\sqrt{1-xx}} + \frac{bs}{1-2xx} \frac{\sqrt{1+3xx}}{1-xx}, \text{ multi-}$$

ply Extreams and Means together of this Analogy and cancel bb , and it becomes this Equation,

$$\frac{x}{1-2xx \sqrt{1-xx}} = \frac{x}{\sqrt{1-xx}} + \frac{s}{1-2xx} \frac{\sqrt{1+3xx}}{1-xx},$$

multiply both Parts of the Equation by $1-2xx \sqrt{1-xx}$ and there comes out $x = x - 2x^3 + s \sqrt{1+3xx}$ $\therefore 2x^3 = s \sqrt{1+3xx}$, and consequently $4x^6 - 3ssxx = ss$. Q. E. I. R. F.

Otherwise,

Put x and y to express the Sine and Cosine of the Angle BAC or CAE , and call AC 1; then will $x = BC = CD$. And *Lastly*, put s and q to express the Sine and Cosine of the given Angle DAE , then $2yx$ and $y^2 - x^2$ will be the Sine and Cosine of the Angle BAE ; $qy^2 - qx^2 + 2syx$ be the Sine and Cosine of the Angle ADC . And *Lastly*, $qx - sy$ is that of CAD , then as Sine Angle $ADC : AC ::$ Sine Angle $CAD : CD$, that is $qy^2 - qx^2 + 2syx : 1 :: qx - sy : x$, and by multiplying Extreams and Means $qxy^2 - qx^3 + 2syx^2 = qx - sy$, and by Transposition $2syx^2 + sy = qx + qx^3 - qxy^2 = 2qx^3$, and by squaring both Sides of the Equation, and writing $1-x^2$ instead of y^2 &c. we get $-4x^6 + 3ssx^2 + ss = 0$, or $-2 \times 2x^3 \times 2x^3 \times 2x^3 + 3 \times 2s^3 \times 2x^3 + 2 \times 2s^3 = 0$. But $2x^3$ is the versed Sine of the Angle BAE , which call v , and $2s^3$ is the versed Sine of 30 Degrees = the double of the Angle DAE , which call p ; then the

H

Equa.

Equation will become $-2v^3 + 3pv + 2p = 0$. Hence

$$v = \sqrt[3]{\frac{1}{2}p + \frac{1}{2}p \sqrt{1 - \frac{1}{2}p}} + \sqrt[3]{\frac{1}{2}p - \frac{1}{2}p \sqrt{1 - \frac{1}{2}p}}$$

$$= 64041\frac{3}{4}c. \text{ hence } CB \text{ or } CD = 65,36264; AD =$$

$$161,73382 \quad AE = 264,87441; BE = 247,16331;$$

and the Area of the Triangle $ABE = 1176:3:18$.
Q. E. I.

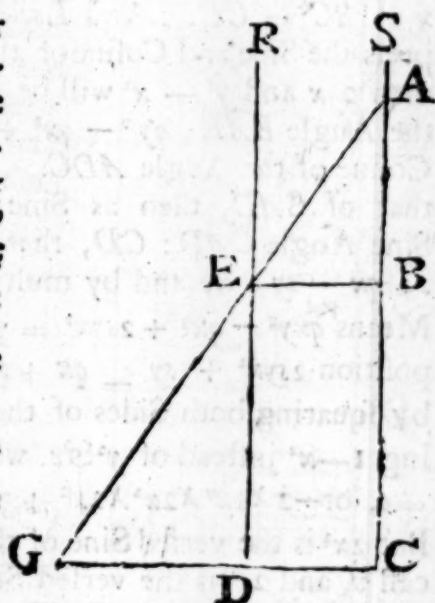
PROB. XXXII.

By Mr. JOHN WATTS.

Let REBS represent a Chimney whose Width EB, (c.) and Height ED, (b.) from the Mantle-Tree to the Floor is given, its required to find the longest Pole, as GA, that can possible be put up this Chimney.

SOLUTION.

PUT x and y for the Sine and Co-sine of the Angle DGE Radius = 1. Then per Trigonometry, as $x : b :: 1 : \frac{b}{x} = GE$, and $y : c :: 1 : \frac{c}{y} = EA$. Hence $\frac{b}{x} + \frac{c}{y} = GA$, which per Question must be a Maximum, and therefore its Fluxion



$$-\frac{\dot{x}b}{x^2} - \frac{\dot{y}c}{y^2} = 0. \text{ But per 47. c. 1. } x^2 + y^2 = 1$$

which put into *Fluxions*, makes $2xx + 2yy = 0 \therefore$

$\frac{\dot{x}x}{y} = -\dot{y}$, which substituted for $-\dot{y}$ above, makes

$$-\frac{\dot{x}b}{x^2} + \frac{\dot{x}xc}{y^2} = 0 \therefore -\frac{b}{x^2} + \frac{xc}{y^2} = 0 \therefore x^3c = by^3$$

$\therefore \frac{x^3}{y^3} = \frac{b}{c} =$ the Cube of the required Tangent of the Angle *DGE*. Q. E. I.

If *z* be put for *AB*, then because $x : z :: y : c$ we have $cx = zy \therefore \frac{x}{y} = \frac{z}{c} \therefore \left(\frac{x^3}{y^3} = \right) \frac{z^3}{c^3} = \frac{b}{c} \therefore z^3 = bc^2$, which is the Cube of *AB*. Q. E. L



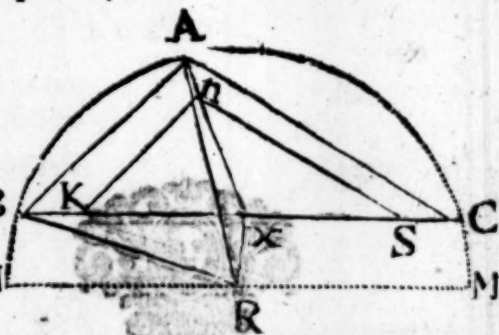
P R O B. XXXIII.

Proposed by Mr. JOHN TURNER.

In an oblique-angled plain Triangle, there is given the Angle opposite to the Base, or longest Side equal to $107^{\circ}.00'$; and a Line drawn from the said angular Point to the Middle of the Base, is known to be 27 Poles, also the Angle which the said bisecting Line makes with the Base is $75^{\circ}.00'$, required the Sides of the Triangle.

The SOLUTION is by Mr. R. G.

CONSTRUCTION, assume $Bx =$ any Number of equal Parts at Pleasure; suppose 48. and make $xC = xB$. Then upon BC describe an Arc of a Circle BAC , capable of containing B an Angle of $107^{\circ}.00'$ by the 33d Proposition



of the 3d of *Euclid*. Then make the Angle $BxA = 75^{\circ}.00'$ and draw the Line xA , and then AB , AC . Upon xA set off $xn = 27$. and draw the Lines nK , nS parallel to BA , CA ; so is the Triangle KnS the Triangle required.

Calculation, draw thro' the Centre R of the Arch, the Diameter MN , parallel to the Base BC of the Triangle, join the Points B and R , and from R let the Line Rx fall perpendicular to the Base

Base BC ; then because BC is a Chord parallel to the Diameter Mn , and Rx perpendicular thereto, Rx shall bisect the Base BC in x . Now the Arch BAC , wherein the given Angle BAC is contained, is equal to the Semicircle NAM , Minus the Sum of the Arches BN and CM ; and so the Angle BAC stands upon a Semicircle, Plus the Sum of the Arches BN and CM ; but from the Elements of Geometry (*vide Euclid*, P. 20. lib. 3.) it is known, that the Quantity of an Angle made in the Periphery of a Circle, is measured by half the Arch upon which it stands; consequently the Angle BAC is equal to $\frac{180 + BN + CM}{2}$; but because of the Parallelism

of the Lines NM and BC , the Arches BN and CM

are equal; and so $\frac{180 + 2Bn}{2} = 90 + Bn$ is =

the Angle $BAC = 107^{\circ}.00'$ and $Bn = BAC - 90 = 107^{\circ} - 90^{\circ} = 17^{\circ}.00'$. But the Arch Bn measures the Central-Angle BRn ; and because the Angles BRn and RBx are alternate Angles made by two Parallels, they are equal, and consequently the Angle RBx is $= 17^{\circ}.00'$ whence because of the right Angle at x , the Angle BRx is $= 73^{\circ}.00'$ wherefore having assumed Bx of any Number of Parts. of a homogenial Denomination, to those wherein the given Line Ax was measured (suppose 48) and the Angles RBx , BRx being known $= 17^{\circ}.00'$ and $73^{\circ}.00'$ in the right-angled Triangle BRx , we may find $BR = 50,2$ Poles, and $Rx = 14,7$. But $RA = RB$, as being Radii of the same Circle, and the Angle $AxR = 165^{\circ}.00'$ ($= 90^{\circ}$ more the Angle BxA). Hence in the Triangle AxR we have sufficient to find $Ax = 36,0$. Now the Triangles BxA , Kxn being similar, as $Ax : xB :: nx : xK = 36$ Poles; consequently $KS = 72$, $Kn = 39,0$, $Sn = 50,3$.

ALGE-

ALGEBRAIC SOLUTION.

PUT $x = Kx = xS$, $s = xn$, and let s and q be put for the Sine and Cosine of the Angle Sxn , then will s and $-q$ express the Sine and Cosine of Kxn . And *Lastly*, let t , express the Tangent of the given Angle KnS , then *per Theorem*, p. 12. l. 3. $\frac{sx}{a+qx} = \text{Tangent of the Angle } Knx$ and $\frac{sx}{a-qx} = \text{Tangent of the Angle } xnS$. Hence *per Theorem* 5. p. 4. find the Tangent corresponding to the Sum of the Angles Knx and xnS , viz.

$$\frac{2sax}{a^2 - q^2x^2 - s^2x^2} = \frac{2sax}{a^2 - x^2} = t \text{ (the Tangent of the Angle } KnS) \text{ which Equation ordered, gives } x =$$

$$\frac{aX: \sqrt{t^2 + s^2} - s}{t}, \text{ or as } t : a :: \sqrt{t^2 + s^2} - s : x$$

$= Kx$ the Semi-Base. Q. E. I.

P R O B. XXXIV.

By Mr. JOHN BADDER.

In the annex'd Triangle EGF, are given EH = a, HF = b, HG = c, and the Angle EGF = any Number of Degrees whatsoever, to find all the other Parts of the Triangle?

SOLUTION.

PUT x and y to express the Sine and Cosine of the Angle FHG , then will x and $-y$, be the Sine and Cosine of the Angle EHG , and *per Theo-*

rem

rem, p. 11. l. 16.

$$\frac{x^b}{c-yb} = T = \text{Tan-}$$

gent of the Angle HGF , and
per Theorem, p.

$$12. l. 6. \frac{x^2}{c+ya}$$

$= t = \text{Tangent of the Angle } EHG$. But having
the Tangent of the Angle EGH , and that of HGF ,

we may per Theorem 5. p. 4. find $\frac{T+t}{1-Tt} =$ (after
a due Reduction, and a Substitution of 1 for x^2+y^2)

$$\frac{x^b c + xac}{c^2 - ybc + yac - ba} = n, \text{ the Tangent of the Angle}$$

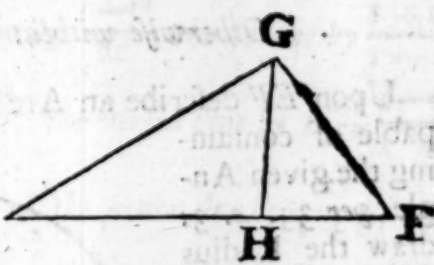
EGF , which Equation multiplied out of Fractions,
makes $x^b c + xac = nc^2 - nybc + nyac - nba$. Put

$$\frac{2t}{1+t^2} \text{ for } x, \text{ and } \frac{1-t^2}{1+t^2} \text{ for } y. \text{ See p. 38. l. 18. re-}$$

duce the Equation and it will come out

$$\left. \begin{array}{l} +anc \\ +anb \\ -cnb \\ -ccn \end{array} \right\} t^2 + \left. \begin{array}{l} +2bc \\ +2ac \end{array} \right\} t = \left. \begin{array}{l} +anc \\ -anb \\ -cnb \\ +ccn \end{array} \right\} \text{ which being}$$

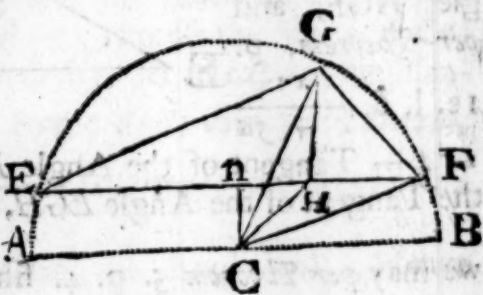
solved will give the Value of t , the Tangent of half
the Angle FHG . Q. E. I.



Other-

Otherwise without ALGEBRA.

Upon EF describe an Arc of a Circle EGF , capable of containing the given Angle *per* 33. *e.* 3. draw the Radius $CF = CG$, join the Points C and E , H with a right Line, and draw the Diameter AB .



parallel to EF . And Lastly, draw the Perpendicular Cn from the Centre C , to the given Base EF ; then (because the Angle CFn may be shewn by the Argument made use of in *Problem* 33. p. 53. l. 21, and 22. to be always = to the Difference betwixt the given Angle EGF and 90 Degrees) we have in the Triangle CnF , all the Angles and Leg nF given, to find Cn , and $CF = CG$; then in the Triangle CnH

is given Cn and $nH = \frac{a-b}{2}$, to find *per* 47. *e.* 1.

CH , and then *per* Trigonometry the Angle CHn , which taken from the Angle CHG , which may be found from the Values of CH , HG and CG , leaves the Angle EHG , then there will be given EH and HG with the included Angle EHG , to find EG , &c. Q. E. D.

The Reader may observe from the Solution of this, and *Problem* 24, 25, 26, 29, and 30. the great Use

of the Expressions $\frac{2t}{1+t^2} = x$, and $\frac{1-t^2}{1+t^2} = y$, in

keeping Equations clear of Surds: We shewed him in *Problem* 23. their Investigation, and shall here just hint to him, that the Tangent is ever expressed by

$\frac{2t}{1+t^2}$, the Cotangent by $\frac{1-t^2}{2t}$, the Secant by $\frac{1+t^2}{1-t^2}$

the Cofecant by $\frac{1+t^2}{2t}$, the versed Sine by $1 - \frac{1-t^2}{1+t^2}$

i. e. the versed Sine of the acute Arc by $\frac{2t^2}{1+t^2}$, and

of the Obtuse by $\frac{2}{1+t^2}$; the Sine of the double Arc by

$\frac{4t-4t^3}{1+t^2|^2}$, it's Cofine by $\frac{6t^2-t^4-1}{1+t^2|^2}$, the Tangent (of

the double Arc) by $\frac{4t-4t^3}{6t^2-t^4-1}$, its Cotangent by

$\frac{6t^2-t^4-1}{4t-4t^3}$; the versed Sine of the double Arc

$2 \frac{X1-t^2}{1+t^2}|^2$, if its Double be less than (90°) a Qua-

drant, and $\frac{8t^2}{1+t^2|^2}$ if greater. All these Things, be-

longing to the Arc whose Sine is x , and Cofine y , are easily found from the common Properties and Proportions of Sines, Tangents, &c. and are chiefly valuable upon account of their Rationality, whereby they keep the Process wherein they are used from the Incumbrance of Surd Quantities. We may perhaps make some further Use of them in the following Part of our Work: But if we do not, the Reader will scarce grudge the Expence of this small Scrap of Paper wherein we have set them down. And now having said enough to enable our Reader (with a little Practice) to give the most concise and elegant Solutions to any Problem, either in Geometry or plain Trigonometry, we shall desist giving any more Examples of this Kind, and make a Transition to Spherics,

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SPHERICAL TRIGONOMETRY.

DEFINITIONS.

1. **I**F a Sphere or Globe be cut by a Plane passing thro' its Center, the common Section will be a Circle, whose Radius will be the Radius of the Globe, and its Circumference will encompass the Surface of the Globe: Such a Circle is called a *great* Circle.

2. But if a Sphere be cut by a Plane not passing thro' its Center, the Section will be a Circle, the *Radii* of which will be less than those of the Sphere; such a Circle is called a *lesser* Circle: Or thus, if we conceive a Circle drawn upon a Globe, and the Globe cut thro' that Circle into two Parts; if the cutting Plane passes thro' the Center of the Sphere, and so divides it into two equal Parts, that Circle is called a *great* Circle; but if the cutting Plane passes beside the Center, and so divides the Sphere into unequal Parts, the Circle is said to be a *lesser* Circle.

It follows then, that of the Circles of the Sphere, the Meridians, the Equator, the Ecliptic, the Horizon, the Circles of Longitude, and the vertical Circles, are *great* Circles; and the Tropics, polar Circles, Circles of Celestial and Terrestrial Latitudes, Circles of Altitudes, and in general, the Secondaries of any *great* Circle, are *lesser* Circles. And besides these more useful, and commonly known Circles of each Kind, which are here mentioned only as Instances of the Difference of one Sort from the other, there may be conceived an

infinite Multitude of both Species, drawn in different Positions to one another, as different Occasions shall require : But all these innumerable *great* Circles, as having each a *Radius* equal to that of the Sphere, will, in the same Sphere, be equal to one another ; whereas of the *lesser* Circles, tho' some few, under particular Circumstances, may happen to agree in Magnitude ; yet nothing is more evident, than that others may dissent from the Size of one another in any imaginable Proportion.

Upon Account of this Inequality of *Radii*, and Diversity of Magnitude among the *lesser* Circles, they are not at all used in *Trigonometry*; but *Spherical Trigonometry* is wholly concerned about the equal *great* Circles of the same Sphere ; and so,

3. A Spherical Triangle is that whose Sides are Arches or Parts of three *great* Circles of the Sphere, or Globe ; or a Triangle made by the joining together, upon a Globe, three Parts of one and the same *great* Circle, as a plain Triangle is made by joining together three straight Lines.

This is a spherical *Triangle* ; and *Spherical Trigonometry* is that Science which teaches, from three given Parts of a spherical Triangle, whether Sides, Angles, or a Mixture of both together, in some Cases, with the Affection of some of the unknown Parts, to find, not the real Lengths of the Sides (as in *Plain Trigonometry*) but the Measures in circular Notation (*i. e.* in Degrees, Minutes, &c.) of the other Sides and Angles.

If the Design of *Spherical Trigonometry* were to calculate the precise Lengths of the Sides of spheric Triangles, it would be requisite, not only that the Sides should be Portions of *great* Circles of the same Sphere, but also that the Diameter or *Radius* of the Sphere should be known, that so, when the Number of Degrees contained in any Arch was known, the Arch itself might be expressed in com-

mon

mon Measures: But as this Science rests in the Knowledge of the Number of Degrees, Minutes, &c. contained in any Arch, without enquiring its Length, it is only necessary, that the Parts of any Triangle of this Kind belong to *great* Circles of the same Sphere, that so, by the Equality of the Magnitudes of the whole, the Lengths of similar Parts of the same Triangle may be the same; a Degree in one Part of the Triangle equal in Length to a Degree in another, &c. Wherefore as all Spheres are similar Figures, and so the Degrees, &c. of one correspondent to those of another, the *Radius* of the Sphere from whence all spherical Triangles are taken, is supposed to be Unity, that so the several Parts may be the better adapted to the Tables of Sines, Tangents, &c. which are all calculated to that *Radius*.

4. A spheric Angle is the Inclination of the Arches of two *great* Circles of the Sphere, meeting in a Point towards one another, and is the same with the Inclination of the Planes of those Circles which constitute the Angle.

5. If you suppose a Point any where upon the Surface of a Sphere, and from this Point conceive a Line drawn thro' the Center of the Globe to the opposite Surface, the Line so drawn is a Diameter to the Globe, and the two Points just mentioned upon the Surface, will lie diametrically opposite; then if you set one Foot of a Pair of Compasses (fit to describe Arches on a convex Superficies) in one of these Points upon the Surface, and extend the other half-way to its Opposite, and with that Interval strike a Circle quite round the Globe, the Circle thus described shall be a *great* Circle, and the two opposite Points in the spheric Superficies its two *Poles*. Hence the *Poles* of any *great* Circle, are those two opposite Points upon the spheric Superficies, from whence if a right Line be drawn thro'

thro' the Center of the Globe, it shall be perpendicular to the Plane of that *great* Circle. If another *great* Circle be supposed drawn thro' the *Poles* of any given *great* Circle, the *Poles* of the given *great* Circle shall divide the Periphery of that so drawn into two Semicircles; and therefore, as all Circles are divided into 360° , the two *Poles* lie 180° distant from one another; and since the Circle lies in the Mid-way between its two *Poles*, it lies 90° from either *Pole*. Hence the *Pole* of any *great* Circle lies a Quadrant, or 90° , from the Circle, measured upon the Surface of the Sphere.

PRINCIPLES.

1. If two *great* Circles intersect one another in any particular Point of the Surface of a Globe, and be both continued from that Point either Way, they will, for a while, recede farther and farther from one another, till they both become Quadrants, when their Aperture will be the greatest; then approaching towards one another, in the same Manner as they before parted, they will unite and cross one another again in a Point diametrically opposite to the first: For all the *great* Circles of the same Sphere are equal, and any two equal Circles whose Centers coincide, but their Planes make an Angle with one another, will cross one another in two opposite Points.

Hence it follows, that every Side of a Spherical Triangle is less than a Semicircle; for if one were a Semicircle, the other two would become but one continued Arch, whence one of the Angles vanishing, the Essence of a Triangle would be lost.

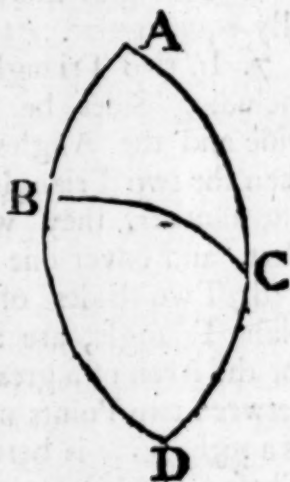
2. If from the angular Point of any Spherical Angle, as from a Pole, a *great* Circle be described, the Arch of this *great* Circle contained between the two Arches which form the Angle, continued till they be Quadrants, is the Measure of that

that Angle; or in other Words, the Measure of a Spherical Angle, is Part of a Circle described upon the angular Point as a Center, and intercepted between the two Legs which form the Angle, continued till they become Quadrants. Now as this Arch, which is said to measure the Spheric Angle, lies 90° from the angular Point, it must consequently lie in the Mid-way between the first and second Intersections of the Arches which form the Angle, and so, as it may be conceived to be described as well from one of these Points of Intersection as the other, it indifferently measures either Angle.

3. If an Arch of a great Circle intersect another in any Point, it makes the Sum of the Angles on the same Side the Arch it intersects equal to 180° , and the vertical Angles equal, as in right Lines.

4. From the two last Principles it follows, that if any two Sides, AB , AC of a Spherical Triangle (ABC) are extended till they intersect one another again in D , the Angle BDC shall be equal to the Angle BAC , the Angle BCD equal the Supplement to 180° of the Angle ACB , and the Angle CBD equal the Supplement to 180° , or a Semicircle of the Angle ABC . Also since (by *Princip. 1st*) the Points A and D are diametrically opposite, and so ABD and ACD each Semicircles, the Side CD will be the Supplement to 180° of the Side AC , and the Side BD of the Side AB ; wherefore the Triangle BDC and ABC have the Side BC and the Angles BAC and BDC common, and the other Parts in one Triangle, the Supplements of the like Parts in the other.

Whence



Whence the Triangles ABC and DBC are called supplemental Triangles in respect to one another; and since the Sine of any Arch, and of its Supplement to 180° , are expressed by the same Number, it will often happen that while we intend solving one Triangle (suppose ABC), we shall in reality find some of the Parts of the supplemental Triangle (BCD). This ought to be observed.

5. In spherical, as in plain Triangles, equal Sides subtend equal Angles; Isosceles Triangles have two equal Angles opposite to the equal Sides; Equilateral Triangles have their Angles equal, and the greater Sides and Angles mutually subtend one another. Also in two Triangles, if the Legs are equal, the Angles they include will be greater or less as the Bases are.

6. Triangles mutually equilateral are also mutually equiangular. *Et contra*,

7. In two Triangles, if one Angle and the two including Sides be respectively equal, or if one Side and the Angles adjacent be severally equal, then the two Triangles are equal; for if laid upon one another, they would perfectly concur in every Part, and cover one another.

8. Two Sides of a Spherical, as well as of a plain Triangle, are always greater than a Third; for the Arch of a great Circle is the nearest Distance between two Points upon the Surface of a Sphere, as a right Line is between two Points upon a Plane. These three last Properties are demonstrated as the like Properties of a plain Triangle are.

9. The Sum of the three Sides of any spherical Triangle (ABC) is less than a Circle; that is, $AB + BC + AC$, (see Fig. last) is less than $ACD + ABD$.

For the Side BC is less than $DB + DC$ (by the last) and adding $AB + AC$ to both Sides, $BC + AB + AC$, the Sum of the Sides of the Triangle ABC

ABC is less than $DB + DC + AC + AB$, or the two Semicircles ABD and ACD .

10. In any spherical Triangle, if the Sum of the Legs, including the vertical Angle, be greater than a Semicircle, the internal Angle at the Base is greater than the opposite external Angle, and the Sum of the two internal Angles at the Base is greater than the two right Angles: If the Sum of the Legs be equal to a Semicircle, then the internal Angle at the Base is equal to the outward and opposite Angle, and the two internal Angles at the Base make a Semicircle. And *Lastly*, If the Sum of the Legs be less than a Semicircle, then the internal Angle at the Base is less than the outward and opposite Angle, and the Sum of the two internal Angles less than a Semicircle.

Hence it follows, that in an Isosceles Triangle, if one of the equal Legs be greater, equal to, or less than a Quadrant, the Angle at the Base is accordingly greater, equal to, or less than a right Angle.

11. If one great Circle pass thro' the Poles of another, those Circles will intersect one another at right Angles.

For (by Def. 5th) the Pole of any great Circle lies in a plane Perpendicular to the Plane of the Circle whose Pole it is; and therefore when a Circle passes thro' the Poles of another, its Plane must be perpendicular to the Plane of that thro' whose Poles it passes: But to have their Planes perpendicular (by Def. 4.) is to cross one another at right Angles. The Converse of this is true too; namely, that when two great Circles are Perpendicular to one another, they pass thro' the Poles of each other.

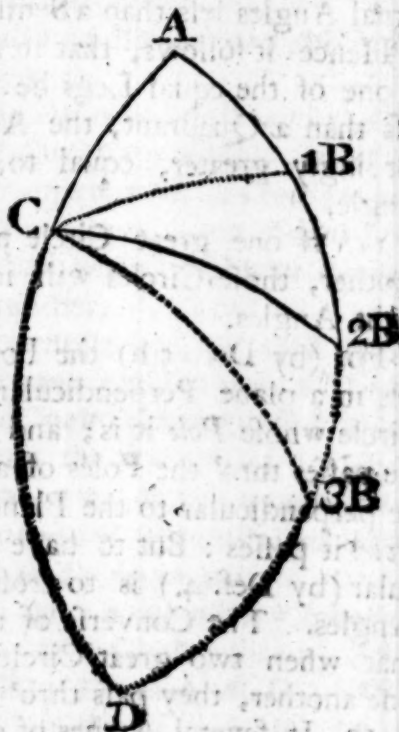
12. If several Arches of great Circles be drawn from any assigned Point of the Surface of a Sphere to another great Circle, the greatest is that which passes thro' the Pole of the Circle, and the shortest

is that which is drawn the opposite Way; the rest are longer or shorter, according to their Vicinity to these two.

13. The Sum of the three Angles of every spheric Triangle is greater than two right Angles, and less than six; and any Angle with the Difference of the other two is less than two right Angles; and lastly, the external Angle is less than the two opposite internal Angles.

14. The Poles of the Sides of any spheric Triangle constitute another Triangle, which we may call supplemental to the first; for the Supplements of the Angles of the second are equal to the Sides of the first, and the Angles of the first to the Supplements of the Sides of the second.

15. In the Triangle ABC , whose Angle A is right, the Angles C and B have the same Affection with their opposite Sides; that is, they are right, acute, or obtuse, according as their opposite Legs are equal to, or less, or greater than a Quadrant. For since the Arch AB is perpendicular upon the Arch AC , it will (by *Princip. 11.*) pass thro' its Pole; therefore if the Arch AB is a Quadrant, the Point B will be the Pole of the Arch AC , and consequently the Angle C will be right. Therefore, when any Leg (as AB) is equal to



to a Quadrant, its opposite Angle (C) is right. But the Arch AB cannot be increased or decreased without augmenting or diminishing the Angle ACB ; therefore as the Angle C is right when AB is a Quadrant, so it must be acute or obtuse as the Side AB is lesser or greater than a Quadrant.

16. In every Rectangular Spherical Triangle, if one or both Legs be Quadrants, the Hypotenuse shall also be a Quadrant; if the Legs are of the same Kind, that is, both acute or both obtuse, the Hypotenuse is less than a Quadrant; but if the Legs are of different Kinds, that is, one acute and the other obtuse, the Hypotenuse is greater than a Quadrant.

For if the Leg AB be a Quadrant, then, from what was said above, the Angle C will be right; but (by *Princip. 5th*) equal Sides subtend equal Angles; therefore since the Leg AB , opposite to the right Angle C , is a Quadrant, the Hypotenuse CB , opposite to the right Angle A , shall be one likewise. The same may be said if the Leg CA be a Quadrant; and therefore if one or both Legs be Quadrants, the Hypotenuse shall be a Quadrant. Again, if the Arch AB be conceiv'd equal to a Quadrant, then a Leg less than a Quadrant cannot reach from A to B , but to some intermediate Point I ; wherefore supposing the Leg AC less than a Quadrant, CI will be the Hypotenuse of the acute legged Triangle CAI , which is manifestly shorter than the quadrantal Hypotenuse CB : So that when the Legs are both less than Quadrants, the Hypotenuse is also less. Prolong the Legs AC and AI till they become Semicircles, and intersect each other in D , then, because the Angles A and D are equal, D will be a right Angle; and so the right-angled Triangle CDI will be supplemental to CAI ; but because CA and AI are each (by the Hypothesis above) acute, CD and DI , their Supplements,

plements, will be obtuse; and so the Triangle CD_1B will have each of its Legs greater than a Quadrant: But the Hypotenuse C_1B , which we just now proved to be less than a Quadrant, is common to both the Triangles AC_1B , CD_1B ; wherefore when the Legs are both obtuse, as well as both acute, the Hypotenuse is less than a Quadrant. Lastly, when one Leg is acute (as AC) and the other C_3B obtuse, then the Hypotenuse C_3B will be evidently longer than the quadrantal Hypotenuse CB , and so be obtuse.

Corollary 1. Right-angled Triangles, having both their Legs Quadrants, have three right Angles; having one Leg acute, and the other obtuse, have, beside the right Angle, one acute and one obtuse Angle; and Triangles having two obtuse Legs, have, together with the right Angle, two obtuse ones.

Corollary 2. Hence the Hypotenuse is acute when the two oblique Angles are of the same kind, and obtuse when they are of a different one.

17. In any oblique angled spherical Triangle, if the Angles at the Base are of the same kind, that is, both acute, or both obtuse, then a Perpendicular let fall from the vertical Angle upon the Base, and dividing the given Triangle into two right angled ones, shall fall within the Triangle, if they are of different kinds without, upon the Base prolonged.

For the Perpendicular will be common to both the right angled Triangles, into which the oblique one is divided; and so (by the 16th *Princip.*) will be of the same Affection with each of the Angles opposite to it, and consequently cannot be opposed to Angles of unlike Kinds; therefore when the Angles at the Base are of the same Kind, the Perpendicular cannot fall without the Triangle; because then it would be opposed by one of the Angles at the Base, and the Supplement of the other to a Semicircle;

micircle; that is, since the Angles at the Base are supposed of the same Kind, and an Angle and its Supplement are, in all Angles (except those which are right) of different Kinds, by Angles of a heterogeneous Kind; neither when the Angles at the Base are of different Affections, can the Perpendicular fall within; by reason, that being then opposed by each of the Angles, it would be opposite to Angles of a different Kind.

THEOREM I.

In any right-angled spherical Triangle, as the Radius is to the Sine of the Hypotenuse, so is the Sine of either of the oblique Angles to the Sine of its opposite Leg. *Et contra,*

PREPARATION.

In the Triangle ABC , right-angled at B , continue BC till BF becomes a Quadrant, so shall the Point F be the Pole of the Arch AB : Also continue AB till AE becomes a Quadrant; then describe the Arch FE , having A for its Pole, and the Arch ACL will be a Quadrant, and the Arch LE the Measure of the Angle A . That done, draw LH right upon the Circle ABE , and it will fall perpendicularly upon the common Section GE , and be the Sine of the Arch EL , or of the Angle A ; also LG will be the Radius, or Sine of the right Angle B . Farther, draw thro' the Point C a Plane parallel to the Plane ELF , and like it perpendicular upon the Plane ABE . This Plane will cut the Plane BCF perpendicular upon ABE , and their common Section Cy will be right upon the same Plane, and consequently parallel to the Line HL , and the Sine of the Arch BC . In like manner Iy and GH , CI and LG are parallel, being the common Sections of the two parallel Planes with a third; and as LG is found perpendicular to GA ,

IC

IC will be perpendicular to GA , and so be the Sine of the Arch AC .

DEMONSTRATION.

The three Sides of the Triangle LHG , being respectively parallel to the three Sides of the Triangle CLy , the two Triangles are equiangular and similar; and therefore as LG , the Sine of the Angle B , is to CI , the Sine of the Hypotenuse AC , so is LH , the Sine of the Angle A , to Cy , the Sine of the opposite Arch BC . Q. E. D.

THEOREM H.

In any right-angled Spherical Triangle, as the Radius : is to the Sine of one Leg, :: so is the Tangent of the oblique Angle adjacent to that Leg : to the Tangent of the other Leg. *Et contra*,

PREPARATION.

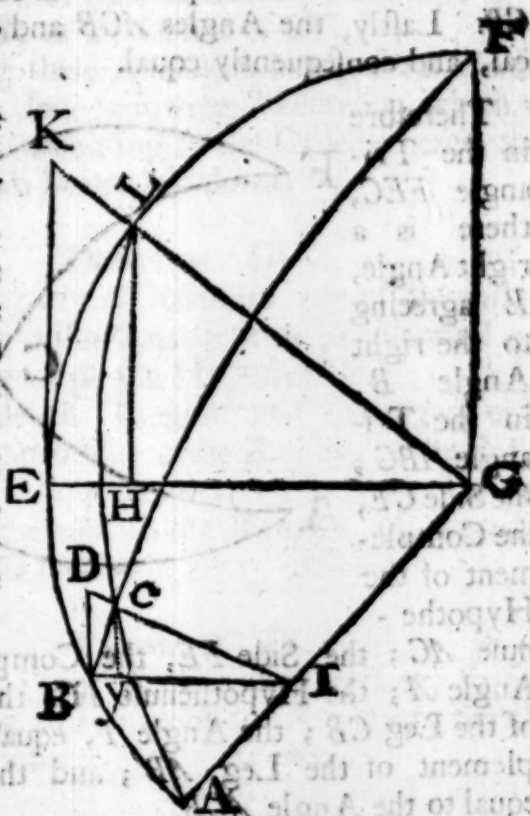
The Arches AB , AC and BC being continued, as before, draw EK , the Tangent of the Arch LE , or of the Angle A , and BD , the Tangent of the Arch BC , and these, as they lie in the Planes of the Circles of which $ELFG$ and BC are Parts, which are perpendicular to the Plane ABE , will also be perpendicular to the said Plane ABE . Farther, thro' the Line BD , draw a Plane parallel to the Plane ELE , and like it perpendicular upon the Plane ABE ; the common Sections DI , BI , will be parallel to the common Sections KG and EG ; and as EG is perpendicular upon FG , BI will also be perpendicular to FG , and will consequently be the Sine of the Arch BA .

DEMONSTRATION.

Since each Side of the Triangle BDI is parallel to each Side of the Triangle KEG ; the two Triangles BDI and KEG are similar; and therefore,

as

The Force of these Demon- strations will be the more easily apprehended, if instead of de- lineating the annex'd Scheme in Plano, its Parts are cut in Paper or PASTE- board, and put together in a proper Man- ner, with the Sines and Tan- gents concern- ed drawn with Ink in their right Places.

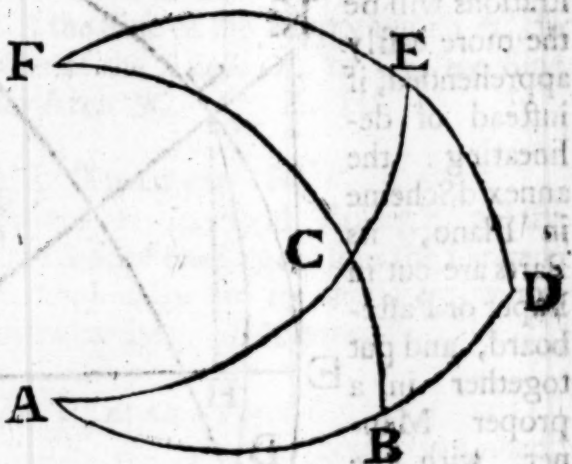


In order to investigate the other Theorems we shall have occasion for, the Solution of the several Cases of right angled sphæric Triangles, let the Sides AB , AC and CB of the Triangle ABC , right-angled at B , be prolonged, so that AD , AE and CF may be each Quadrants; and from the Pole A describe the Arch FED , then (by *Princip. 11.*) the Arch FED will be perpendicular both to the Arch ACE and ABD , and so the Angles E and D , as well as B , will be right. Moreover, because the Angles B and D are each right, the Arch FED , as well as FCB (by *Princip. 17.*) will be a Quadrant.

Far-

Farther, because ACE and ABD are each Quadrants, DB , which (since AD is a Quadrant) is the Complement of AB , will be the Measure of the Angle F . Also, since ACE and FCB are each Quadrants, CE will be the Complement of AC , and CF of CB . Lastly, the Angles ACB and FCE are vertical, and consequently equal.

Therefore in the Triangle FEC , there is a right Angle, E agreeing to the right Angle B , in the Triangle ABC ; the Side CE , the Complement of the Hypothe-



nuse AC ; the Side FE , the Complement of the Angle A ; the Hypothenufe FC , the Complement of the Leg CB ; the Angle F , equal to the Complement of the Leg AB ; and the Angle FCE , equal to the Angle ACB .

Whence first, by Theorem 2d, As Radius : is to the Sine of FE , :: so is the Tangent of the Angle F to the Tangent of EC ; that is, as Radius : to the Cosine of the Angle A , :: so is the Cotangent of AB : to the Cotangent of AC .

2dly, by Theorem 1st. As Radius : is to the Sine of FC , :: so is the Sine of the Angle C : to the Sine of FE . That is, as Radius : is to the Cosine of CB , :: so is the Sine of the Angle C : to the Cosine of the Angle A .

3. By Theorem 2d. As Radius : is to the Sine of CE , :: so is the Tangent of the Angle C : to the Tangent of FE ; that is, as Radius : is to the Co-

sine

Sine of AC :: so is the Tangent of Angle C : to the Cotangent of Angle A .

4. By Theorem 1st, As Radius : to the Sine of FC :: so is the Sine of Angle F : to the Sine of CE ; that is, as Radius : to the Cosine of BC :: so is the Cosine of AB : to the Cosine of AC .

By putting these Analogies into general Terms, we have the four following Theorems, which we shall number according to the Order wherein they succeed the two before-laid down.

THEOREM III.

As Radius : to the Cosine of one of the oblique Angles, :: so is the Cotangent of its adjacent Leg : to the Cotangent of the Hypothenufe.

Or, because the Tangent and Cotangent of an Arch are Reciprocals ; as the Radius : to the Cosine of one of the oblique Angles, :: so is the Tangent of the Hypothenufe : to the Tangent of the adjacent Leg. *Et contra.*

THEOREM IV.

As Radius : to the Cosine of either Leg, :: so is the Sine of its adjacent oblique Angle : to the Cosine of its opposite Angle. *Et contra.*

THEOREM V.

As Radius : to the Cosine of the Hypothenufe, :: so is the Tangent of one oblique Angle : to the Cotangent of the other. *Et contra.*

THEOREM VI.

As Radius : to the Cosine of one Leg, :: so is the Cosine of the other : to the Cosine of the Hypothenufe. *Et contra.*

These six Theorems and their Inverses solve all the Cases of Right-angled spherical Triangles, which tho' usually made sixteen in Number, as there can be but six Varieties of Data, the rest being in the *Quesita*; we shall, to comport with our design'd Brevity, reduce them to that Number only; but at the same Time give Examples in each of our Cases of finding all the unknown Parts, and so in Fact make our six involve the whole sixteen Cases of other Authors.

It is supposed in the Demonstrations of the preceding Theorems, that all the Sides and Angles, except the right one, are acute; at least nothing is said to the contrary; but if any of the Parts (retaining the right Angle) are in any Case obtuse, the Theorems hold, by substituting the Supplements of the obtuse Parts, instead of the Parts themselves (see *Princip. 4.*) The Reader is desired to note this once for all.

As we shall have frequent Occasion in the ensuing Parts of this Treatise to make an algebraical Use of Spherics, in the Solution of difficult Problems, we shall not only give an Analogy, but also an Equation for finding all the unknown Parts in each Case: In order to which let the Right-angled spherical Triangle *ABC*, be furnished with Letters, as in the Figure; in which it is to be observed that we continually express the Sines of Arches or Angles by the small Letters placed or set on the Inside of those Arches or Angles, and their Cosines by those placed on the Outside.

it answers in the Tables, are what we seek after, or its Supplement to a Semicircle.

Upon this Account, those Theorems, where they can be had, are to be esteemed for Trigonometrical Purposes, which are composed of Tangents, Cotangents, Secants, and versed Sines, beyond those which are made up wholly of Sines.

It is farther to be noted, that Radius being 1, (as we all along suppose) $\frac{e}{f}$ will represent the Tangent of the Arch AB , $\frac{f}{e}$ its Cotangent, $\frac{1}{f}$ its Secant, $\frac{s}{z}$ the Tangent of the Angle A , $\frac{z}{s}$ its Cotangent, and $\frac{1}{z}$ its Secant, and so of all other Arches or Angles,



The Resolution of the Six Cases of Right-angled Spherical Triangles, by the Canon of Artificial Sines, Tangents, &c. and of some by the Natural Canon, in a new and compendious Method.

PREPARATORY to the Logarithmic Solution of the several Cases, we shall say nothing; as taking it for granted, that our Reader is acquainted with the Tables of Logarithms, Sines, Tangents, &c. and their Use; but as we believe our other Method of Solution to be entirely new, it will be requisite, as briefly as may be, to open the Springs from whence it flows, and accommodate it to Practice by proper Rules and Precepts.

1st then, Put A and B to express the Sine and Cosine respectively of any Number of Degrees, G and H for the Sine and Cosine of another Number of Degrees, s and C for the Sine and Cosine of the Sum, and D and d for the Sine and Cosine of the Difference of those two Arches; then by *Theorem* 2d, p. 3. if both Arches be acute, and their Sum be less than a Quadrant, $AG - BH$, or if their Sum exceeds a Quadrant, $BH - AG$ will be $= C$; or if one be acute, and the other obtuse, then $AG + BH = C$. Also, by *Theorem* 4. p. 3. When both Arches are acute $AG + BH = d$, when one is acute, and the other obtuse, and their Difference less than a Quadrant, $AG - BH = d$; and when one is acute, and the other obtuse, but their Difference greater than a Quadrant, $BH - AG$ will be $= d$. Therefore, by adding the several Expressions of d and C together, when both Arches are acute,
and

and their Sum less than a Quadrant, $2AG$ is = $C+d$, or $AG = \frac{C+d}{2}$; when their Sum exceeds a Quadrant, $2BH$ is = $C+d$, or $BH = \frac{C+d}{2}$, and when one Arch is acute, and the other obtuse, and their Difference less than a Quadrant, $2AG = C+d$, and $AG = \frac{C+d}{2}$; and lastly, when their Difference is more than a Quadrant, $2BH$ is = $C+d$, and $BH = \frac{C+d}{2}$. And by subtracting in each Case the Values of C and d from one another, it appears that when both Arches are acute, and their Sum less than a Quadrant, $2BH$ is = $d-C$, and $BH = \frac{d-C}{2}$; when their Sum is greater than a Quadrant, $2AG = d-C$ and $AG = \frac{d-C}{2}$; when one Arch is acute, and the other obtuse, and their Difference less than a Quadrant, $2BH = C-d$ and $BH = \frac{C-d}{2}$; and finally, when their Difference exceeds a Quadrant, $2AG = C-d$, and $AG = \frac{C-d}{2}$.

From comparing these Expressions together, it will be manifest, that if adding C and d together in any particular Case, produce the Rectangle of the Sines of two Arches, then taking the Difference of C and d in that Case, will give the Rectangle of the Cosines of those Arches, and the contrary. Wherefore, by this Observation, and from the Equations for each Case, we have these two Rules.

RULE I.

RULE I.

When two Arches are both acute, and their Sum less than a Quadrant, or one acute, and the other obtuse, add the Cosine of the Sum, and of the Difference of the two Arches together; half the Sum is the Rectangle of the Cosines, and half the Difference of the Cosine of the Sum and Cosine of the Difference, is the Rectangle of the Sines of those two Arches.

RULE II.

When both Arches are acute, but their Sum above a Quadrant, or when one Arch is acute, and the other obtuse, and their Difference less than a Quadrant; add together the Cosines of the Sum, and Difference of the given Arches; half the resulting Sum is the Rectangle of the Sines, and half the Difference between the Cosine of the Sum and Cosine of the Difference, is the Rectangle of the Cosine of those same two Arches.

These Rules are to be particularly valued for this, that they give the Rectangles of the Sines and Cosines of any two Arches together, by an Operation from the same Numbers.

2d, The Letters bearing the same Signification as before, $AH + BG$ (by *Theorem 1.* p. 3.) will be $= S$, and $AH - BG = D$; wherefore by Addition, $2AH$ will be found $= S + D$, or $AH = \frac{S+D}{2}$; and by Subtraction, $2BG = S - D$, or $BG = \frac{S-D}{2}$: Whence

RULE III.

R U L E III.

Add the Sine of the Sum and Sine of the Difference of any two Arches together; half the Sum is the Rectangle of the Sine of the greater into the Cosine of the lesser; and subtract the Sines of the Sum and Difference from one another, half the Remainder is the Rectangle of the Sine of the lesser Arch into the Cosine of the greater.

These Rules will take in all the Cases of spherical Triangles, except two, which they will solve in near as expeditious a Manner as the Logarithms, and to all the Exactness desirable. Our ingenious Friend Mr. WILLIAM MILES of *Worcester*, has compleated the Thing in all its Varieties, and invented a whole *Sinical Arithmetick*, built upon these Principles, and would have communicated the Rules necessary to supply the Deficiencies of Ours; but as it would have taken up a good deal of Room to explain and illustrate them, and as the Logarithmic Method, after all, is most convenient in Practice, we thought proper to omit them; and so have only subjoined an Operation, by our own Rules, just now explain'd, to the Cases they will reach.



CASE I.

CASE I.

Given the Leg $AB = 67^\circ : 30'$, and the adjacent Angle $CAB = 50^\circ : 15'$; to find the other Parts.

N.B. The last Figure will be referred to through, all the Cases.

1. For the Leg BC .

ALGEBRAICALLY, by Theorem 2d.
 as Radius ($= 1$) : to the Sine of AB
 ($= e$) :: so is the Tangent of the Angle A
 ($= \frac{s}{z}$) : to $\frac{s}{z}e =$ the Tangent of BC ; that is,
 $\frac{s}{z}e$ is $= \frac{e}{d}$.

By the Artificial Canon.

From the same Theorem, as Radius 10.

To the Sine of the given Leg AB } 9.9656153
 $= 67^\circ : 30'$

So is the Tangent of the Angle } 10.0800375
 $A = 50^\circ : 15'$

To the Tangent of the required } 10.0456528
 Leg $BC = 48^\circ : 00'$

By Princip. 16. The Leg BC will be acute or obtuse, according to the Affection of the given Angle A . The same Thing may also be inferred

from the Equation $(\frac{s}{z}e = \frac{e}{d})$, for while $\frac{s}{z}$ is Af-

firmative, $\frac{e}{d}$ will be so too; but when $\frac{s}{z}$ is Nega-

tive, $\frac{e}{d}$ will also be Negative.

2. To find the Hypothenuſe AC.

ALGEBRAICALLY. By Theorem 3. as Radius ($=1$) : to the Coſine of the Angle $A (=z)$:: ſo is the Cotangent of $AB (= \frac{f}{e})$: to $\frac{f}{e}z = \frac{b}{a}$, the Cotangent of the Hypothenuſe AC.

By the Logarithms.

From the ſame Theorem, as Radius 10.

To the Coſine of the Angle $A = 50^\circ : 15'$ } 9.8057991
 So is the Cotangent of the Leg }
 $AB = 67^\circ : 30'$ } 9.6172245

To the Cotangent of the Hypothe- }
 nuſe $AC = 75^\circ : 10'$ } 9.4230236

By Princip. 16. The Leg CB and the Angle A are of the ſame Affection; and by Princip. 17. when the two Legs are of the ſame kind, the Hypothenuſe is acute, and when of a different, obtuſe; therefore, when the Leg AB and Angle A are both acute, or both obtuſe, the Hypothenuſe is acute, but when they have a different Affection, it is obtuſe. The ſame Thing alſo flows from the Equation

$(\frac{f}{e}z = \frac{b}{a})$; for while $\frac{f}{e}$ and z are each Affir-

mative, or each Negative, the Product $\frac{f}{e}z = \frac{b}{a}$ will be Affirmative; but when they have different Signs, Negative.

To find the Angle C.

ALGEBRAICALLY. By Theorem 4. as Radius ($=1$) : to the Coſine of $AB (=f)$:: ſo is the Sine of the Angle $A (=s)$: to the Coſine of the Angle C ; conſequently $fs = q$. By

By Logarithms.

From the same *Theorem*, As Radius 10.

To the Cosine of the Leg $AB = \left. \begin{array}{l} 67^\circ 30' \end{array} \right\} 9.5828397$

So is the Sine of the Angle $A = \left. \begin{array}{l} 50^\circ : 15' \end{array} \right\} 9.8858370$

To the Cosine of the Angle $C = \left. \begin{array}{l} 72^\circ : 53' \end{array} \right\} 9.4686767$

By the natural Canon.

$AB = 67^\circ : 30'$
Angle $A = 50 : 15$

Sum $\underline{117 : 45}$ Sup. $62^\circ 15'$ Nat. Sine .8849876

Diff. $\underline{17 : 15}$ Nat. Sine .2965416

Since we here want the Rectangle of the Cosine of a greater Arch into the Sine of a lesser, by Rule 3d, the Difference whose Half is $\left. \begin{array}{l} .5884460 \text{ is } = 2f, \\ .2942230 = f \end{array} \right\}$
the Cosine of Angle C ; to which answers in the Tables $72^\circ : 53'$ as above.

The Angle C (by *Prin.* 16.) will be acute or obtuse, as the opposite Leg AB is; which is also manifest from the Equation $fs = q$, for while f is Affirmative $fs = q$ will be so too, but when it is Negative, q will be Negative also.

If the Leg CB and its adjacent Angle C had been given to find the other Parts, the Analogies and Observations here used would have served.

C A S E II.

Given the Angle $A = 50^\circ : 15'$, and its opposite Leg $CB = 48^\circ : 00'$; to find the other Parts.

1. To find the Leg AB .

ALGEBRAICALLY. By the Reverse of Theorem 2. as the Tangent of the Angle $A \left(= \frac{s}{z} \right)$: to the Tangent of the opposite Leg $CB \left(= \frac{c}{d} \right) ::$ so is $Rad. (=1)$: to $\frac{c}{d} \frac{z}{s} = e =$ the Sine of AB .

By Logarithms.

By the Reverse of the same Theorem, } 10.0800379
as the Tangent of Angle $A = 50^\circ : 15'$.

To the Tangent of the opposite } 10.0455626
Leg $CB = 48^\circ : 00'$.

So is the Radius 10.

To the Sine of the Leg $AB = \}$ 9.9655247
 $67^\circ : 29'$.

2. To find the Hypothenuse AC .

ALGEBRAICALLY. By the Reverse of Theorem 1. as the Sine of the Angle $A (=s)$: to the Sine of the opposite Leg $CB (=c) ::$ so is $Radius (=1)$: to $\frac{c}{s} = a =$ the Sine of the Hypothenuse AC .

By the Logarithms.

As the Sine of the Angle $A = 50^\circ : 15'$ 9.8858370

To the Sine of its opposite Leg } 9.8710725
 $CB = 48^\circ : 00'$

So is the Radius 10.

To the Sine of the Hypothenuse } 9.9852365
 $AC = 75^\circ : 9'$

3. 10

3. To find the Angle C.

ALGEBRAICALLY. By the Reverse of Theorem 4. as the Cosine of the Leg CB ($= d$) : to the Radius ($= 1$) :: so is (z) the Cosine of the Angle A : to $\frac{z}{2} = p$ = the Sine of the Angle C .

By Logarithms.

As the Cosine of the Leg CB	}	9.8255109
$= 48^\circ : 00'$		
To the Radius		10.
So is the Cosine of the Angle A	}	9.8057991
$= 50^\circ : 15'$		

To the Sine of the Angle $C =$ } 9.9802882
 $72^\circ : 53'$

All the three Varieties of this Case are ambiguous ; for since (by *Prin.* 16.) the Angle A and the Leg BC are ever of the same Kind ; knowing that A and BC are both acute or both obtuse, does not in the least inform us of the Affection of any of the other Parts of the Triangle ; therefore without the Affection being given of another Part beside the Data, it can't be known whether the *Quesita* be acute or obtuse ; and therefore whether the Thing found be what was required, or its Supplement to a Semicircle.

If the Angle C , and its opposite Leg AB had been given, the other Parts might have been found by the Analogies, &c. here laid down.

CASE III.

CASE III.

Given the Hypothenuſe $AC = 75^\circ : 10'$, and one of the oblique Angles $A = 50^\circ : 15'$, to find the other Parts of the Triangle.

1. To find the Leg AB .

ALGEBRAICALLY. By the latter Expreſſion of Theorem 3. as Radius $(=1)$: to the Coſine $(=z)$ of the Angle A : ſo is $\left(\frac{a}{b}\right)$ the Tangent of the Hypothenuſe : to $\frac{a}{b} z = \frac{e}{f}$ the Tangent of the Leg AB .

By the artificial Canon.

From the ſame Theorem as Radius 10.

To the Coſine of the Angle A	}	9.8057991
$= 50^\circ : 15'$		
So is the Tangent of the Hypo-	}	10.5770265
thenuſe $= 75^\circ : 10'$		

To the Tangent of the Leg AB	}	10.3828256
$= 67^\circ : 30'$		

From Prin. 16 and 17. the Leg AB will be acute when the Hypothenuſe and the Angle A have the ſame Affection, and obtuſe when their Affection is different. The ſame Thing is manifeſt from the Equation $\frac{a}{b} z = \frac{e}{f}$; for while $\frac{a}{b}$ and z have both the ſame Sign, $\frac{e}{f}$ will be Affirmative, but when they have different ones, Negative.

2. To find the Leg CB .

ALGEBRAICALLY. By Theorem 1. as Radius $(=1)$: to the Sine of the Hypothenuſe

nuse ($=a$) :: fo is (s) the Sine of the Angle A
 : to $sa = c$ = the Sine of the Leg CB . This
 Equation being wholly composed of Sines, shews
 this Variety to be ambiguous.

By the Logarithms.

By the same Theorem as Radius $101 : 100 ::$
 To the Sine of the Hypothenufe } 9.9852803
 $AC = 75^\circ : 10'$ }
 So is the Sine of the oblique An- } 9.8858370
 gle $A = 50^\circ : 15'$ }

To the Sine of the Leg $BC =$ } 9.8711173
 $48^\circ : 00'$ }

By the natural Canon.

$AC = 75 : 10$
 $Ang. A = 50 : 15$

Sum $125 : 25$ whose Nat. Cofine is $.5795183$

Diff. $24 : 55$ Nat. Cofine $.9069215$

Since both Archés are acute, and their
 Sum greater than a Quadrant (by Rule 2.) } $1.4864398 = 2sa$
 adding we have

Half of which is the natural Sine of BC , } $.7432199 = sa$
 viz. }

To which answers in the Tables $48^\circ : 00'$, as above.

3. To find the Angle C .

ALGEBRAICALLY. By Theorem 5.
 as Radius ($= 1$) : to (b) the Cofine of the Hy-
 pothenufe :: fo is ($\frac{s}{z}$) the Tangent of the Angle

$A : \text{to } \frac{s}{z} b = \frac{q}{p} = \text{the Cofine of the Angle } C.$

By

By the Logarithms.

From the same Theorem as Radius 10.

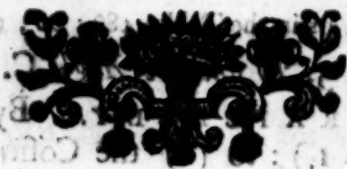
To the Cosine of the Hypothenuſe $AC = 75^\circ : 10'$ } 9.4082539

So is the Tangent of the Angle A } 10.0800379
 $= 50^\circ : 15'$

To the Cotangent of the Angle C } 9.4882918
 $= 72^\circ : 53'$

It is easily deducible from the 16 and 17 *Prin.* or from the Equation $\frac{b}{z} = \frac{q}{p}$, that when the given Angle and the Hypothenuſe are of the ſame Affection, the other Angle (C) ſhall be acute, but when of a different, obtuſe.

The ſame Analogies, Equations, and Obſervations would have found the other Parts, from the Hypothenuſe and Angle C being given.



C A S E

CASE IV.

Given the Legs $AB = 67^\circ : 30'$, and $BC = 48^\circ : 00'$, to find the other Parts.

1. To find the Hypothenuſe.

ALGEBRAICALLY. By Theorem 6.
As Radius ($= 1$) : to (f) the Coſine of $AB ::$ ſo is (d) the Coſine of $CB : to $fd = b$, the Coſine of the Hypothenuſe.$

By the Logarithms,

From the ſame Theorem, as Radius 10.

To the Coſine of the Leg $AB = \left. \begin{array}{l} 67^\circ 30' \end{array} \right\} 9.5828397$

So is the Coſine of the other Leg $\left. \begin{array}{l} CB = 48^\circ : 00' \end{array} \right\} 9.8255109$

To the Coſine of the Hypothenuſe $\left. \begin{array}{l} AC = 75^\circ : 10' \end{array} \right\} 9.4083506$

By the natural Canon.

$AB = 67^\circ : 30'$

$BC = 48 : 00$

Sum = 115 : 30 Sup. $64^\circ 30'$ Nat. Coſine .4305111

Diff. 19 : 30 Nat. Coſine .9426415

Since both Arches are acute, and their Sum greater than a Quadrant, by ſubſtracting (as directed in Rule 2d.) we have $\left. \begin{array}{l} \text{Sum} \\ \text{Diff.} \end{array} \right\} .5121304 = 2fd,$

Whoſe half is the natural Coſine of the Hypothenuſe, viz. $\left. \begin{array}{l} \text{Sum} \\ \text{Diff.} \end{array} \right\} .2560652 = fd = b.$

To which answers in the Tables $75^\circ 10'$, as before found.

N

By

By *Prin. 17.* If the two Legs have the same Affection, the Hypothenufe is acute; if different, obtufe. The same Thing may also be feen from the Equation $fd = b$.

2. To find the Angle A.

ALGEBRAICALLY. By the Reverse of *Theorem 2d*, as the Sine of $AB (= e)$: to the Radius $(= 1)$: : fo is the Tangent of $CB (= \frac{e}{d})$ to : $\frac{e}{de} = \frac{1}{z}$ the Tangent of the Angle A.

By *Logarithms.*

As the Sine of $AB = 67^\circ : 30'$ 9.9656153
 To the Radius 10.
 So is the Tangent of the Leg BC } 10.0455626
 $= 48^\circ : 00'$
 To the Tangent of the Angle } 10.6799473
 $A = 50^\circ 15'$

3. To find the Angle C.

ALGEBRAICALLY. By the Inverse of *Theorem 2d*, as the Sine of $BC (= e)$: to the Radius $(= 1)$: : fo is the Tangent of the Leg $AB (= \frac{e}{f})$: to $\frac{e}{fe} = \frac{p}{q}$ the Tangent of the Angle C.

By *Logarithms.*

As the Sine of $BC = 48^\circ : 00'$ 9.8710735
 To the Radius 10.
 So is the Tangent of the Leg AB } 10.3827757
 $= 67^\circ : 30'$
 To the Tangent of the Angle } 10.5117022
 $C = 72^\circ : 53'$

By

By *Princip.* 16. The Angles *A* and *C* will have the same Affection with their opposite Legs *BC* and *AB*.

CASE V.

Given the Hypothenuſe $AC = 75^{\circ} : 10'$, and one Leg $AB = 67^{\circ} : 30'$; to find the other Parts.

1. To find the Leg *BC*.

ALGEBRAICALLY. By the Inverſe of *Theorem* 6. as the Coſine of the Leg *AB* ($= f$) : to the Radius ($= 1$) :: ſo is the Coſine of the Hypothenuſe ($= b$) : to $\frac{b}{f} = d$ = the Coſine of the Leg *BC*.

By the Logarithms.

As the Coſine of the Leg *AB* = } 9.5828397
 $67^{\circ} : 30'$

To the Radius

So is the Coſine of the Hypothe- } 9.4082539
 nuſe = $75^{\circ} : 10'$

To the Coſine of the Leg *BC* = } 9.8254142
 $48^{\circ} : 00'$

By *Princip.* 17. The Leg *BC* will be acute when the Hypothenuſe *AC* and Leg *AB* are of the ſame Kind, and obtuſe when their Affection is contrary.

2. To find the Angle *C*.

ALGEBRAICALLY. By the Reverse of *Theorem* 1. as the Sine of the Hypothenuſe ($= a$) : to the

the Rad. ($= 1$) :: so is the Sine of $AB (= e)$: to $\frac{e}{a} = p$, the Sine of the Angle C . This Equation being made up wholly of Sines, indicates the Ambiguity of this Variety.

By Logarithms.

As the Sine of the Hypothenufe	}	9.9852803
$= 75^\circ : 10'$		
To the Radius		10.
So is the Sine of the Leg AB	}	9.9656153
$= 67^\circ : 30'$		
For the Sine of the Angle C	}	9.9803350
$= 72^\circ : 53'$		

3. To find the Angle C .

ALGEBRAICALLY. By the Reverse of the latter Expression of *Theorem 3*. as the Tangent of the Hypothenufe ($= \frac{a}{b}$) : to the Tangent of AB ($= \frac{e}{f}$) :: so is Radius ($= 1$) : to $\frac{eb}{fa} = z$ = the Cosine of the Angle A .

By the Logarithms.

As the Tangent of the Hypothenufe	}	10.5770265
$= 75^\circ : 10'$		
To the Tangent of the Leg AB	}	10.3827757
$= 67^\circ : 30'$		
So is the Radius		10.
To the Cosine of the Angle A	}	9.8057492
$= 50^\circ : 15'$		

While

While $\frac{e}{f}$ and $\frac{b}{a}$ are both affirmative or both negative, z will be affirmative; but negative when $\frac{e}{f}$ and $\frac{b}{a}$ have different Signs: Therefore when the Hypothenufe and the Leg AB have the same Affection, the Angle A shall be acute, and obtuse when they are of different Kinds. The same Thing may be deduced from the 16th and 17th Principles.

The Equations, Proportions, and Notes would have been the same for finding the other Parts from the Hypothenufe and Leg CB being given.

CASE VI.

Given the two oblique Angles, $A = 50^\circ : 15'$ and $C = 72^\circ : 53'$; to find the other Parts of the Triangle.

1. To find the Hypothenufe AC .

ALGEBRAICALLY. By Theorem 5. inverted, as the Tangent of one Angle, $A \left(= \frac{s}{z} \right)$

: to the Cotangent of the other, $C \left(= \frac{q}{p} \right) ::$ so is

Radius ($= 1$) : to $\frac{qz}{ps} = b$, the Cosine of the Hypothenufe AC .

By

By Logarithms. By the same.

As the Tangent of one oblique }
 Angle (suppose A) = $50^\circ : 15'$ } 10.0800379
 To the Cotangent of the other, }
 $C = 72^\circ : 53'$ } 9.4884924
 So is Radius 10.

To the Cosine of the Hypothe- }
 nuse $AC = 75^\circ : 10'$ } 9.4084545

By Cor. 2. to Prin. 17. the Hypothenuse is acute when the two oblique Angles A and B have the same Affection, and obtuse when they have a different.

2. To find the Leg AB .

ALGEBRAICALLY. By the Reverse of Theorem 4. as the Sine of the Angle A , adjacent to the enquired Leg AB (i.e. s) : to the Cosine of the opposite Angle, C ($= q$) :: so is the Radius ($= 1$) : to $\frac{q}{s} = f$, the Cosine of the Leg AB .

By the Logarithms. By the same.

As the Sine of the adjacent Angle }
 $A = 50^\circ : 15'$ } 9.8858370
 To the Cosine of the opposite }
 Angle $C = 72^\circ : 53'$ } 9.4688173
 So is Radius 10.

To the Cosine of the Leg AB }
 $= 67^\circ : 30'$ } 9.5829803

3. To find the Leg BC .

ALGEBRAICALLY. By the Reverse of Theorem 4. as the Sine of the adjacent Angle C ($= p$) : to the Cosine of the opposite Angle, A ($= z$) :: so is Radius ($= 1$) : to the Cosine of the Leg BC ; that is, $\frac{z}{p}$ is $= d$.

By

By the Logarithmic Canon

As the Sine of the adjacent Angle, $C = 72^\circ : 53'$ } 9.9803250

To the Sine of the opposite Angle, $A = 50^\circ : 15'$ } 9.8057998

So is the Radius, $10.$

To the Cosine of the Leg BC } 9.8254741
 $= 48^\circ : 00'$

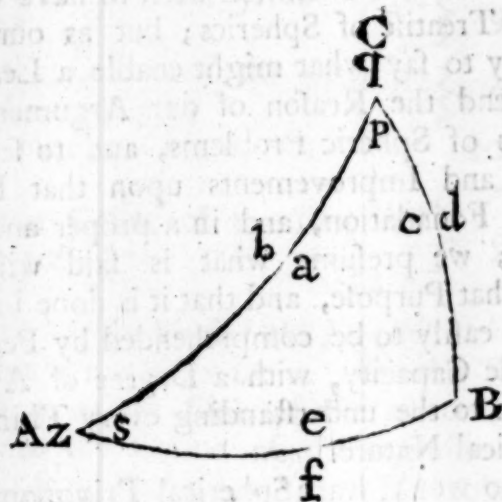
By *Prin. 16.* the Legs AB and BC will be acute or obtuse, according to the Affection of their opposite Angles C and A .

Thus have we shewn the Solution to every Variety of each Case of right-angled spherical Triangles, by the Help of the Six Theorems before laid down, and demonstrated; but it may seem a considerable Defect that we have not also given their Solutions by the Noble *Theorem* invented by the celebrated Lord *Napeir*; and indeed this is such an Omission that we ourselves should have thought unpardonable, had our Intention been to have wrote a compleat Treatise of Spherics; but as our Design was only to say what might enable a Learner to comprehend the Reason of our Arguments in the Solutions of Spheric Problems, and to set our Discoveries and Improvements upon that Head on a solid Foundation, and in a proper and natural Light; we presume what is said will be sufficient to that Purpose, and that it is done in such a Manner as easily to be comprehended by Persons of a tolerable Capacity, with a Degree of Attention requisite to the understanding every Thing of a Mathematical Nature.

Those who would learn Spherical Trigonometry only to make some practical Use thereof, may in this and what follows, pass over the *Algebraical*,
 and

and consider only the *Logarithmic* Solutions; and if such are desirous of seeing a greater Variety of Examples and Methods of Operation, and of the Manner of applying it, we refer them to such Authors as have treated more largely upon the Subject; especially to a most compleat and useful Piece of this Kind shortly to be published by Mr. *Dougharty* senior, of *Worcester*: A Gentleman whose justly acquired Reputation in the World makes his Name a sufficient Recommendation to any of his Works; where he will find Trigonometry *Plain* and *Spherical* handled in the most full, clear, and easy Manner; together with the Projection of the Sphere, and their several Applications to *Astronomy*, *Navigation*, *Dialling*, &c.

We shall just present the Reader with a brief Synopsis of the Solutions to the several Cases of right-angled Triangles, as they have been before laid down, and then proceed to those which are oblique angled.



SYNOPSIS

Cases.

I. A

II. A

III. A

IV

V

A

VI. A

N.

Cosine,

		SYNOPSIS.		
Cases.	Given.	Sought.	Analogies.	Equation.
I.	AB	BC	$As R : s, AB :: s, \text{Ang. } A : t, BC$	$\frac{se}{t} = \frac{c}{d}$
	Ang. A	AC	$As R : cs, \text{Ang. } A : ct, AB : ct, AC$	$\frac{Ax}{b} = \frac{b}{a}$
		Ang. C	$As R : cs, AB :: s, \text{Ang. } A : ts, \text{Ang. } C$	$\frac{t}{s} = \frac{a}{q}$
II.	CB	AB	$As t, \text{Ang. } A : t, CB : R :: R : s, AB$	$\frac{t}{s} = \frac{e}{c}$
	Ang. A	AC	$As t, \text{Ang. } A : s, CB : R :: R : s, AC$	$\frac{t}{s} = \frac{a}{c}$
		Ang. C	$As cs, CB : R :: cs, \text{Ang. } A : s, \text{Ang. } C$	$\frac{E}{p} = \frac{p}{q}$
III.	AC	AB	$As R : cs, \text{Ang. } A : t, AC : t, AB$	$\frac{ax}{b} = \frac{a}{f}$
	Ang. A	CB	$As R : s, AC :: t, \text{Ang. } A : t, BC$	$\frac{Pa}{b} = \frac{c}{q}$
		Ang. C	$As R : cs, AC :: t, \text{Ang. } A : ct, \text{Ang. } C$	$\frac{wb}{p} = \frac{q}{p}$
IV.	AB	AC	$As R : cs, AB :: cs, BC : cs, AC$	$\frac{fd}{b} = \frac{b}{c}$
	CB	Ang. A	$As t, AB : R :: t, BC : t, \text{Ang. } A$	$\frac{c}{ae} = \frac{s}{z}$
		Ang. C	$As t, BC : R :: t, AB : t, \text{Ang. } C$	$\frac{e}{p} = \frac{a}{q}$
V.	AC	BC	$As cs, AB : R :: ts, AC : cs, BC$	$\frac{p}{f} = \frac{d}{a}$
	CB	Ang. C	$As s, AC : R :: s, AB : s, \text{Ang. } C$	$\frac{e}{a} = \frac{p}{q}$
		Ang. A	$As t, AC : t, AB : R :: ts, \text{Ang. } A$	$\frac{cb}{aa} = \frac{z}{x}$
VI.	Ang. A	AC	$As t, \text{Ang. } A : ct, \text{Ang. } C : R : cs, AC$	$\frac{qx}{ps} = \frac{b}{a}$
	Ang. C	AB	$As s, \text{Ang. } A : ts, \text{Ang. } C : R : cs, AB$	$\frac{t}{s} = \frac{f}{a}$
		BC	$As s, \text{Ang. } C : ts, \text{Ang. } A : R : cs, BC$	$\frac{z}{p} = \frac{d}{a}$

N. B. In this Synopsis R stands for Radius ($= 1$), s for Sine, cs for Cosine, t for Tangent, ct for Cotangent, and Ang. for Angle.

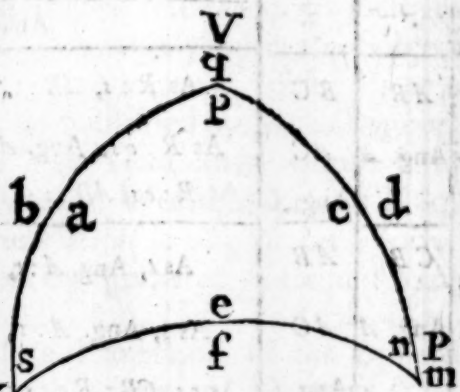


Of

Of oblique-angled Spherical Triangles.

THE Values of the several Letters made use of in the ALGEBRAICAL Demonstrations and Solutions belonging to this

Part of *Trigonometry*; as also those denoting the Sides and Angles of the Triangles used herein, may be learnt from the adjacent *general Scheme*; wherefore we shall take up no Time in an unnecessary Prologue, but begin immediately upon the Theory, and from thence proceed to the Practice.



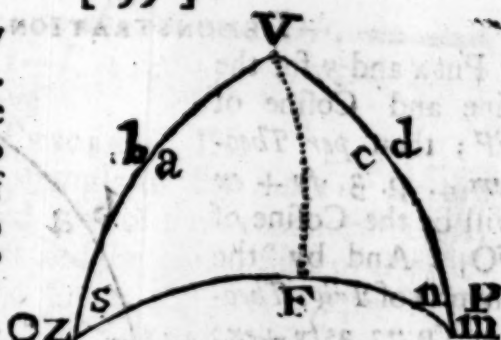
THEOREM VII.

In every Spherical Triangle, whether right or oblique-angled, as the Sine of one Angle : to the Sine of its opposite Side :: so is the Sine of another Angle : to the Sine of its opposite Side. *Et contra.*

That the Proposition before us is true, in such Triangles as have a right Angle, was asserted in *Theorem 1.* which having been demonstrated, needs not be again repeated; and to prove, that it holds also in oblique-angled Triangles, let *OV* be such a Triangle, divided into two right-angled ones by the Perpendicular *VF*, let fall upon the Base *OP*.

Then

Then ist. By
Spheric Theor. 1.
as Radius : to the
Sine of VP :: so
is the Sine of
Angle P : to
 $s, VP \times s, \text{Ang. } P$



$\frac{\text{Rad.}}{\text{Sine of } OZ}$
= the Sine of VF ; again, from the same Theorem, as Radius : to
the Sine of VO :: so is the Sine of the Angle O : to
 $s, VO \times s, \text{Ang. } O$
 $\frac{\text{Rad.}}{\text{Sine of } VF}$ = the Sine of VF ; consequently

$$\frac{s, VP \times s, \text{Ang. } P}{\text{Rad.}} = \frac{s, VO \times s, \text{Ang. } O}{\text{Rad.}}. \text{ Or,}$$

by equal Multiplication, $s, VP \times s, \text{Ang. } P = s, VO \times s, \text{Ang. } O$; wherefore by putting the Equation into Analogy, as $s, \text{Ang. } O$: to s, VP , :: so is $s, \text{Ang. } P$: to s, VO . *Et contra.* Q. E. D.

THEOREM VIII.

In any Spherical Triangle (the Radius being supposed Unity) the Rectangle of the Sines of any two Sides into the Cosine of their contained Angle, being called a first Product, and the Rectangle of the Cosines of the said Sides styled a Second; the Sum of these two Products, when the Angle is acute, and the containing Sides of like Affection, or when the Angle is obtuse, and the Sides one obtuse and the other acute; and their Difference, when the Sides are both of the same Denomination, and their contained Angle obtuse, or when the Sides have different Affections and the Angle is acute, is equal to the Cosine of the third Side.

Or $+df + cem = b$.

fine of the Difference of PK and OP , we have
 $D - cev = b$, $\therefore 1 - D + cev = 1 - b$, which Equations in Words give

THEOREM IX.

Multiply the Rectangle of the Sines of the two Sides into the versed Sine of their contained Angle, and to this Product add the versed Sine of the Difference of the two Sides, the Sum is equal to the versed Sine of the third Side required.

Or THEOREM X.

From the versed Sine of the Sum of the Sides subtract the versed Sine of their Difference, the Rectangle of Half the Remainder, and versed Sine of the given Angle being added to the versed Sine of the Difference of the Sides, gives the versed Sine of the Side sought.

From which is deduced THEOREM. XI.

When all the Sides of any Spherical Triangle is given, to find an Angle.

From the versed Sine of the Side opposite to the Angle sought, subtract the versed Sine of the Difference of the other two Sides; this Remainder divided by the Excess of the versed Sine of the Sum of the two Sides (including the Angle sought) above the versed Sine of their Difference, gives twice the versed Sine of the required Angle.

In these two last Theorems, since there is no Quantity which can from affirmative become negative, there will not be any Ambiguity in the Solution of Problems wrought by them.

THEOREM XII.

In all Spherical Triangles, where there are given two Sides and their contained Angle, to find one of the other Angles. Divide the Sine of the Side adjacent to the required Angle by the Sine of the given Angle, and multiply the Quotient by the Cotangent of the Side opposite to the Angle sought, reserving the Product: Also, multiply the Tangent of the given Angle into the Cosine of the interja-cent,

cent Side; then the *Sum* of these two Products, when the Side opposite to the enquired Angle is obtuse, and the given Angle and interjacent Sine have a like Affection, or when the opposite Side is acute, and the interjacent Side and given Angle have different Affections; and their *Difference* when the opposite Side to the required Angle is acute, and the interjacent Side and given Angle have a like Affection, or when the Side opposing the Angle sought is obtuse, and the given Angle and interjacent Side are of different Denominations, is the Cotangent of the Angle desired.

$$\text{Or } \frac{+de + fcm}{cn} = \frac{z}{s}.$$

DEMONSTRATION.

By *Trig. Theorem 8.* $fb + aez = d$, (see the last Fig.) and $fd + cem = b$, and by exterminating b , we have $f^2d + fcm + aez = d$. But by *Trig. The-*

orem 7. As $s : c :: n : \frac{nc}{s} = a$; hence by writing

$\frac{nc}{s}$ for a in the last Equation, $f^2d + fcm + \frac{cnez}{s} = d$, which multiplied by s , after Transpo-

sition gives $fcems + cnez = ds - f^2ds = dse^2$; which divided by e becomes $fcms + cnz =$

dse , or $cnz = des - fcms$: Ergo $\frac{z}{s} = \frac{de - fcm}{nc}$

$= \frac{de}{cn} - \frac{m}{n}f =$ the Cotangent of the Angle O .

Q. E. D.

COROLLARY

Since $\frac{de - fcm}{nc}$ is $= \frac{z}{s}$, if v be put $=$ the versed Sine of the given Angle (P), m will be equal to $1 - v$, and therefore by writing this Value instead

of m , the Equation will become $\frac{de - fc + fcw}{nc} = \frac{x}{s}$,
 but $de - fc$ is $= D$ the Sine of the *Difference*
 of the two given Sides ; consequently $\frac{D + fcw}{nc}$
 is the Cotangent of the required Angle ; whence we
 have.

THEOREM XIII.

Multiply together the Sine of the Side opposite
 to the enquired Angle, the Cosine of that adjacent
 thereto, and the versed Sine of the known Angle :
 When the Side adjacent to the Angle required is
 acute, add this Product to the Sine of the *Difference*,
 but when obtuse, take the Difference between this
 Product and the Sine of the *Difference* of the known
 Sides ; and divide this Sum or Difference by the
 Rectangle under the Sines of the given Angle, and
 of the Sine opposite to that required ; the Quotient
 shall give the Cotangent of the Angle sought.

THEOREM XIV.

In any Spherical Triangle, where there are three
 Angles given to find a Side, multiply together the
 Cosines of the Angles adjacent to the enquired Side ;
 if these two Angles are of *like* Affection, and the
 Third *acute*, or if *unlike*, and the Third *obtuse*, add
 this Product and the Cosine of the third Angle to-
 gether ; but if the two Angles lying at the Extre-
 mities of the required Side be of the *same* Denomi-
 nation, and the third be *obtuse*, or of *different* De-
 nominations, and the third *acute* ; take the Diffe-
 rence between this Rectangle and the Cosine of the
 other Angle ; the Sum or Difference in all Cases di-
 vided by the Rectangle of the Sines of the adjacent
 Angles, shall give the Cosine of the required Side.

Or $\frac{+zm + q}{sn} = f.$

DEMON-

DEMONSTRATION.

By Trig. Theorem 12. $\frac{de-fcm}{nc}$ is $= \frac{z}{s}$ and $\frac{fc-dem}{ne} = \frac{q}{p}$. By the first of these Equations d is $= \frac{ncz + fcms}{se}$, and by the second $= \frac{fcp - qne}{emp}$; consequently $\frac{ncz + fcms}{se} = \frac{fcp - qne}{emp}$, or, in Integers, $nmpcz + fcmsp = fcp - qne$, or $nmpcz + qne = fcp - fcmsp$; but $1 - m^2$ is $= n^2$, and so $fcp - fcmsp = fcspn^2$, ergo $nmpcz + qne = fcspn^2$; which Equation divided by n gives $czmp + qes = fcpsn$. But by Trig. Theorem 7. as $s : c :: p : e$; ergo $se = cp$, which being put for se in the foregoing Equation, makes $czmp + qcp = fcpsn$, which divided by cp , gives $zm + q = fsn$; whence $f = \frac{zm + q}{sn}$.

Thus the Theorem stands when all the Angles are acute, and by changing the Signs of the Terms wherein the Cosines are concerned from affirmative to negative, as the Nature of the Case requires, It will be adapted to any designed Circumstances, and will evidently demonstrate the verbal Theorem in all its Parts to be true. Q. E. D.

The Truth of this Theorem might have been deduced in a more usual and concise Manner from Trig. Theorem 8. by Help of Prin. 14. but we thought this the more clear; and, upon account of some Disputes concerning this Matter (*vide* Dr. Keill's *Euclid* with Mr. Gunn's Remarks) since this is the more uncontrovertable Method of the two, we chose to make use of it; and the rather, because we imagine it to be new, and to comport better with our Algebraical Manner of Procedure.

From

From these Theorems which have been already demonstrated, may be deduced such as follow, and that by a bare Reduction, which as they are chiefly serviceable to *Algebraical* Purposes, we shall not stand either to put into Words or to demonstrate; but just hinting their Origin, shall leave to the Reader the Pleasure of analytically investigating them:

THEOREM XV.

$$m = \frac{b-df}{ce}. \text{ From Theorem 8.}$$

THEOREM XVI.

$$\frac{d}{c} = \frac{fms + nx}{se}. \text{ From Theorem 12.}$$

Or THEOREM XVII.

$\frac{d}{c} = \frac{S-msv}{se}$ wherein S is $= ms + nx$ = the Sine of the Sum of the Angles P and O , and v = the versed Sine of the adjacent Side (PO). This is deduced from the last.

THEOREM XVIII.

$1-q = V + snv$ gain'd. From the 14th.

Where V = versed Sine of the Supplement of the Sum of the Angles P and O , and v = the versed Sine of the adjacent Side (PO).

THEOREM XIX.

$$\frac{x}{s} = \frac{qVp^2 - n^2e^2 - fnm}{n^2 - p^2} = \frac{bpq - fnm}{n^2 - p^2}$$

THEOREM XX.

$$\frac{d}{c} = \frac{fVc^2 - p^2a^2 - qab}{c^2 - a^2} = \frac{mef - qab}{c^2 - a^2}$$

THEOREM XXI.

$\frac{m}{n} = \frac{szf - pqd}{z^2 - q^2}$ where p and q is put for the Sine and Cosine of the Angle PVO .

THEOREM XXII.

$\frac{f}{z} = \frac{abz - cdm}{a^2 - c^2}$ where a and b is put for the Sine and Cosine of the Side VO .

Note, These four last *Theorems* are easily obtain'd from the 8th and 12th.

'Tis thro' all these *Theorems* supposed, that the several Parts of the *Data* are acute; but if any are otherwise, the Signs of those Parts of the Equation where the Cosines of the obtuse Parts are concerned must be changed, as we have before directed.

Having finished the Theory of the *Algebraical* Part, we shall now in a few Words dispatch that which concerns the common Methods of Solution. And first then, *The given oblique-angled Triangle* (in the Cases falling without the Reach of *Trig. Theorem 7.* except in the two last Cases, where it will be ineffectual) must be divided, by a Perpendicular let-fall from the End of a Side given, adjacent to an Angle given, opposite to that Angle, and falling upon some Point of the Side required, or its Continuation, and, if it may be, opposite to the Angle sought, into two right-angled Triangles, in such a Manner, that two of the given Parts may lie whole and entire in one of the Triangles, and the Third entire in the other, or else Part in one and Part in the other Triangle.

Then beginning in that Triangle of the two right-angled ones, wherein there are two Things given, find (by the *Theorems* belonging to right-angled Triangles) that Part of the Third given Thing which lies within this first Triangle; or if no Part of it lie therein, that Part lying in the same Sine,

or

or belonging to the same Angle, whereto the said third Part belongs.

Then taking the Sum or Difference, as the Case requires, of the Third Part, and the Side or Angle thus found, you will have one Thing known in the second Triangle; from whence, and the Data in the First, the Thing sought may be found by Help of some of the Four following Theorems.

THEOREM XXIII.

A Perpendicular being let fall from the vertical Angle of an oblique-angled Spherical Triangle, upon the Base, it shall be, as the Cotangent of one Leg : is to the Cosine of that Part of the vertical Angle lying next to it :: so is the Cotangent of the other Leg : to the Cosine of that Part of the vertical Angle adjacent to it; that is, as $ct, VP : cs, Ang. FVP :: ct, OV : cs, Ang. OVF$. Et contra.

DEMONSTRATION.

By the Reverse of the first Expression of Trig. Theorem 3. as $cs, Ang. FVP : 1 (= Rad.) ::$ so is

$ct, VP : to \frac{ct, VP}{cs, Ang. FVP} = t, VF$. Again, from

the same, as $cs, Ang. OVF : 1 (= Rad.) :: ct, OV$

: to $\frac{ct, OV}{cs, Ang. OVF} = the t, VF, ergo \frac{ct, VP}{cs, Ang. FVP}$

$= \frac{ct, OV}{cs, Ang. OVF}$, or $ct, VP \times cs, Ang. OVF = ct,$

$OV \times cs, Ang. FVP$; or in Analogy, as $ct, VP ; cs, Ang. FVP :: ct, OV : cs, Ang. OVF$. Q. E. D.

THEOREM XXIV.

Things being as before; as the Cosine of one Leg : to the Cosine of the Segment of the Base next to it :: so is the Cosine of the other Leg : to the Cosine of the Segment of the Base adjacent thereto,

thereto, viz. as $cs, VP : cs, FP :: cs, OV : cs, OF$.
Et contra.

DEMONSTRATION.

From *Trig. Theorem 6.* as $cs, FP : cs, VP :: Rad.$
 $(= 1) : \frac{cs, VP}{cs, FP} =$ the Cosine of VF . And by
the same as $cs, OF : cs, OV :: Rad. (= 1) : \frac{cs, OV}{cs, OF}$
 $=$ the Cosine of VF , consequently $\frac{cs, VP}{cs, FP}$
 $= \frac{cs, OV}{cs, OF}$, and $cs, VP \times cs, OF = cs, OV \times cs, FP$;
wherefore in Proportionals, as $cs, VP : cs, FP ::$
 $cs, OV : cs, OF$. Q. E. D.

THEOREM XXV.

As the Cotangent of one Angle at the Base : is
to the Sine of the Segment of the Base next to it :
so is the Cotangent of the other Angle at the Base :
to the Sine of its adjacent Segment of the Base, viz.
as $ct, Ang. P : s, PF :: ct, Ang. O : s, OF$. *Et*
contra.

DEMONSTRATION.

By *Trig. Theorem 2.* as $1 (= Rad.) : \text{the Sine of}$
 $FP :: t, Ang. P : s, FP \times t, Ang. P = t$, of VF .
And from the same, as $1 (= Rad.) : s, OF :: t,$
 $Ang. O : s, OF \times t, Ang. O = t$, of VF : Hence $s,$
 $FP \times t, Ang. P = s, OF \times t, Ang. O$, and so in
Analogy, as $t, Ang. P : s, OF :: t, Ang. O : s,$
 FP ; but the Tangent and Cotangent of an Arch
are Reciprocals; therefore, as $ct, Ang. P : s, FP ::$
 $ct, Ang. O : s, FO$. Q. E. D.

THEOREM XXVI.

As the Cosine of one Angle at the Base : is to the
Sine of that Part of the vertical Angle lying towards
it : so is the Cosine of the other Angle at the Base
: to the Sine of that Part of the vertical Angle ly-
ing

ing on the same Side the Perpendicular with it;
that is, as cs , $Ang. P : s$, $Ang. FVP :: cs$, $Ang. O :$
 s , $Ang. OVF$.

DEMONSTRATION.

From *Trig. Theorem 4.* as s , $Ang. FVP : cs$, $Ang.$

$P :: 1 (= Rad.) : \frac{cs, Ang. P}{s, Ang. FVP} = cs$, of VF ; and

by the same, as s , $Ang. OVF : cs$, $Ang. O :: 1$

$(= Rad.) : \frac{cs, Ang. O}{s, Ang. OVF} = cs$, of VF ; wherefore

$\frac{cs, Ang. P}{s, Ang. FVP} = \frac{cs, Ang. O}{s, Ang. OVF}$, and cs , $Ang. P \times s$,

$Ang. OVF = cs$, $Ang. O \times s$, $Ang. FVP$; therefore

in Proportionals, as cs , $Ang. P : s$, $Ang. FVP ::$
 cs , $Ang. O : s$, $Ang. OVF$. Q. E. D.

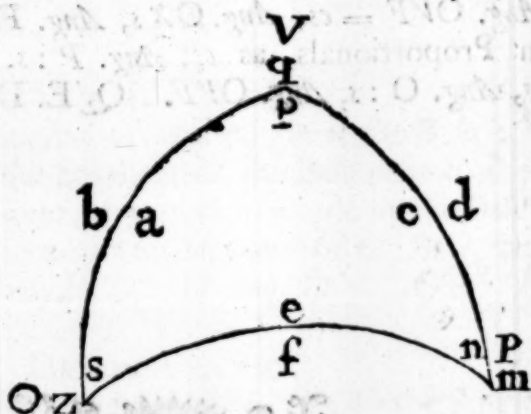


The Solution of the Twelve Cases of oblique-angled Spherical Triangles.

CASE I.

Given two Angles, $VOP = 86^\circ : 11'$ and $OPV = 80^\circ : 12'$, with a Side $OV = 80^\circ : 00'$, opposite to one of them, to find the Side VP opposite to the other.

ALGEBRAICALLY
from Theorem
7. as $n : a :: s$
 $\frac{as}{n} = c =$ the
Sine of VP .



Logarithmically by the same Theorem.

As the Sine of the Angle OPV } 0.0063840
= $80^\circ : 12'$. * *Comp. Arithm.*

To the Sine of the Side $OV =$ } 9.9933515
 $80^\circ : 00'$

So is the Sine of the Angle VOP } 9.9990357
= $86^\circ : 11'$

To the Sine of the required Side } 9.9987712
 $VP = 85^\circ : 42'$

* The Complement Arithmetical of any Logarithm is found by beginning at the left Hand, and taking each Figure out of Nine ; but the last towards the right Hand out of Ten ; or by begin-
Since

Since by *Princip. 5.* the greater Side subtends the greater Angle; it may sometimes (as in this Example) be from thence determined, whether the Arch found be the required Side or its Supplement; or rather whether the Arch found in the Tables (which may be called the acute Value) or its Supplement to 180° (which may be stiled the obtuse Value) may one singly, or either indifferently, be taken for the Side required; but 'tis oftentimes ambiguous; to know when it is so and when not, the following Rule may be used:

R U L E

To the acute Value of the Side sought, and also to its obtuse one, add the given Side; if each of these Sums have the same Affection with the Sum of the two given Angles; then may either the acute or obtuse Value of the required Side be admitted, and therefore it is ambiguous: But when only one of these Sums has the same Affection with the Sum of the given Angles, then that Value only of the Side sought (whether obtuse or acute) which produces a Sum having the same Denomination with the Sum of the said Angles, is to be admitted; and so there is, in that Case, no Ambiguity.

ing at the right Hand, and subtracting the whole from 10,000: it is very useful in Trigonometrical Analogies, where the Radius is not concerned; because it turns the Subtraction of the first from the Sum of the Second and Third, into an Addition together of all the Three.



C A S E

C A S E II.

Given two Sides, $OV = 80 : 00'$ and $VP = 85 : 42'$, with an Angle, $OPV = 80^\circ : 12'$ opposite to one of them; to find the Angle $VO P$, opposite to the other. (*Vide Fig. the last*).

ALGEBRAICALLY. By the Inversion of Theorem 7. as $a : n :: c : \frac{cn}{a} = s =$ the Sine of the required *Ang. VOP*.

By the Logarithms, from the Inverse of the same Theorem.

As the Sine of the Side $OV = \left. \begin{array}{l} 80^\circ : 00'. \text{ Comp. Arith.} \end{array} \right\} 0.0066485$

Is to the Sine of its opposite Angle $OPV = 80^\circ : 12' \left. \begin{array}{l} \end{array} \right\} 9.9936160$

So is the Sine of the Side $VP = \left. \begin{array}{l} 85^\circ : 42' \end{array} \right\} 9.9987758$

To the Sine of the required *Ang. VOP* $= 86^\circ : 11' \left. \begin{array}{l} \end{array} \right\} 9.9990403$

In this Case, as in the last, it may sometimes be known, by Help of *Princ. 5.* whether the Angle sought be the Arch taken out of the Tables, or its Supplement to a Semicircle, tho' it frequently is ambiguous; when it is so, and when not, may be discovered by the Rule subjoined to the last Case, making a proper Change in the Words, to accommodate it to the present *Data*.

C A S E

C A S E III.

*Given two Sides, and their contained Angle,
to find the Third Side.*

EXAMPLE I.

LET the Side OP be $= 72^\circ : 00'$, $OV = 52^\circ : 00'$ and their contained Angle $VOP = 49^\circ : 00'$, to find the third Side VP .

Algebraically. Since the Parts given are all acute, by *Theorem 8.* $aez + bf = d =$ the Cosine of the required Side VP .

By Logarithms, according to the Direction of the same Theorem.

Side $OP = 72^\circ : 00'$ L. s. 9.9782063 L. cs. 9.4899824

Side $OV = 52 : 00$ L. s. 9.8965321 L. cs. 9.7893420

A. $VOP = 49 : 00$ L. cs. 9.8169429 Sum 9.2793244

Sum 9.6916813 is N. No. 1902499 = bf .

Answering Nat. Number is .4916786 = aez } Add, because the
Rect. of the Cosines is .1902499 = bf } Data are all of the
same Affection.

Sum .6819285 = the Nat. Cos. of $VP = 47^\circ : 00'$



Operation, according to the same Theorem, by the Natural Canon.

Side OP = $720 : 00'$

Side OV = $52 : 00$

Sum $124 : 00$ Nat. Cofine .5591929

Diff. $20 : 00$ Nat. Cofine .9396926

Sum 1.4988855

half Sum $.7494427 =$ (per Rule 2. p. 79) to $ae =$ the Rect. of the Sines of OP and OV.

Difference $.3804997$

half Difference $.1902498 =$ (per the same Rule) to bf , or the Rect. of the Cof. of OP and OV.

Now $ae (= .7494427)$ the Rectangle of the Sines of the two Sides being multiplied by $(.6560590)$ the Nat. Cofine of $(49^\circ : 00')$ the contained Angle, produces $.4916786$, which added to $bf (= .1902498)$ the Rectangle of the Cofines of the Sides, makes $.6819284$, = the Nat. Cofine of the Side sought; to which answers $47^\circ : 00'$, as before *.

Logarithmic Operation by Theorem 9.

Log. Sine of OP = $72^\circ : 00'$

9.9782063

Log. Sine of OV = $52 : 00$

9.8965321

Log. verf. S. of Ang. VOP = $49 : 00$

9.5364840

Sum

9.4112424

Answering

Nat. Numb.

.2577640

* N. B. The Product of aez may truly be found by Addition and Subtraction in the following Manner, viz.

Side

Side $OP = 72^\circ : 00$

Side $OV = 52 : 00$

The Diff. is $20 : 00$ Nat. V. Sine .0603074

Nat. Numb. just found is .2577640

Sum is the N. V. Sine of $PV = 47^\circ : 00$.3180714

By letting fall a Perpendicular.

The Perpendicular VF being let fall from the Angle V , upon the Base OP , in the right-angled Triangle OPF , are known OP and the Angle POF ;

Wherefore, by Theorem 3.

As Radius

10.

To the Cosine of the Angle O } 9.8169429
 $= 49^\circ : 00'$

So is the Tangent of the Hypo- } 10.1071902
 then use $OP = 52^\circ : 00'$

To the Tangent of $OF = 40^\circ : 01'$ 9.9241331

To $124^\circ =$ Sum of the Sides OP and OV .

Add $49 =$ Ang. POV

Sum 173 N. cs. = .9925462

Diff. 75 N. cs. = .2588190

The Diff. .7337272

And also to $20^\circ =$ their Difference

Add $49 =$ Ang. POV .

Their Sum 69 N. cs. = .3583679

Diff. 29 N. cs. = .8746197

Sum 1.2329876

Then $\frac{1}{4}$ of .7337272 + 1.2329876 = .4916787 = *arz.*

Q 2

Now

Now $OP = 72^\circ : 00'$, made less by $OF = 40^\circ : 01'$, leaves $FP = 31^\circ : 59'$; whence,

By Theorem 24.

As the Cosine of $OF = 40^\circ : 01'$	} 0.1158521
<i>Comp. Arith.</i>	
To the Cosine of $OV = 52 : 00'$	
So is the Cosine of $FP = 31^\circ : 59'$	9.7893420
	9.9284994
To the Cosine of VP required =	} 9.8336935
$47^\circ : 00'$, as before	

EXAMPLE II.

Let the Side OV , in the adjacent Triangle, be $= 71^\circ : 50'$, $VP = 58^\circ : 12'$ and their contained Angle $OVP = 145^\circ : 10'$, to find the third Side OP .

Logarithmic Operation by Theorem 9.

Log. Sine of $VO = 71^\circ 50'$	9.9777938
Log. Sine of $VP = 58 : 12$	9.9293641
Log. Versed Sine $Ang. V = 145 : 10$	10.2602664
Sum	10.1674243
Answering Nat. Number	1.47033622

$VO = 71^\circ : 50'$
 $VP = 58 : 12$

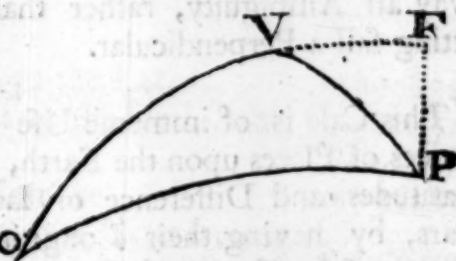
Diff. $13 : 38$	Nat. Versed Sine .0281759
Nat. Numb. found add	1.47033622

The Sum is the Nat. Versed Sine of } 1.4985381
 $119^\circ : 54$

By

By letting fall a Perpendicular.

A Perpendicular being let fall from the Angle P upon the Side OV produced, in the Right-angled Triangle VFP will be



known $VP = 58^\circ : 12'$, and the Angle $FVF =$ the Supplement of the Angle $OVP = 180^\circ - 145^\circ : 10' = 34^\circ : 50'$; wherefore,

By Theorem 3.

As Radius

10.

To the Cosine of $Ang. FVP = 34^\circ : 50'$

9.9142464

So is the Tangent of $VP = 58 : 12$

10.2075899

To the Tangent of $VF = 52^\circ : 56'$

10.1218363

Now VF added to OV , gives $OF = 124^\circ 46'$; and by Theorem 24.

As the Cosine of $VF = 52^\circ : 56'$
Comp. Arith.

0.2198672

Is to the Cosine of $VP = 58 : 12$

9.7217742

So is the Cosine of $OF = 124^\circ 46'$

9.7560544

To the Cosine of the Side $OP =$
 $119^\circ : 54'$

9.6975958

When the two Sides are of the same Kind, and their contained Angle acute, the Side sought is less than a Quadrant; but greater when the Sides are of different Affections, and the Angle obtuse: In other Circumstances, if the Things from whence the Triangle is taken do not inform you of the Affection of the Side sought, the best Way will be to use

Theorem

Theorem the 9th, which is so contrived as to take away all Ambiguity, rather than the Method of letting fall a Perpendicular.

This Case is of immense Use in finding the Distances of Places upon the Earth, by knowing their Latitudes and Difference of Longitude, and of Stars, by having their Longitudes and Latitudes, or right Ascensions and Declinations; and to all these Purposes *Theorem* 9. (which I invented some Years ago) seems best adapted; but above all, this *Theorem* is of excellent Use in calculating the Sun's Altitude at all Hours; for it finds, by a very easy Operation, not only Altitudes at every Hour or Minute of a Day when the Sun has a given Declination, but at the same Time, its Altitude at the same Hour or Minute on the Day when the Sun has the same Declination on the contrary Side of the Equator; so that if any industrious Person have a Mind to compose Tables of the Sun's Altitude for any Latitude, we presume he cannot easily meet with a more expeditious Rule to compute them by than this.

The Rule is, Add together the artificial Cosines of the Latitude of the Place and Declination of the Sun; the Sum call a fixed Number for that Declination, and thereto add the Logarithmic Versed Sine of the Hour from Noon; and find the Natural Number answering to this Sum, which add to the Natural Versed Sine of the Sun's Meridian Zenith Distance on the given Day, and it will give the Natural Versed Sine of his Zenith Distance at the Hour proposed.

Example. Let it be required to find the Altitude of the Sun when he has $23^{\circ}:29'$ North and South Declination, at two and five o'Clock in the Afternoon, or (which is the same) for the Hours of ten and

and Seven in the Morning, in the Latitude ($51^{\circ} 32'$) of London.

Log. Cofine of $51^{\circ} 32$ the Lat. 9.7938317

Log. Cofine of $23 : 29$ the Decl. Sun 9.9624527

Fixed Number 9.7562864

Log. Verfed Sine of the Hour from Noon 30° } 9.1270224

Sum 8.8833068

Answering Nat. Numb. .0764376

Summer Meridian Zenith Distance 28° } .1174624

03' Nat. Verfed Sine }
Nat. Numb. found add .0764376

Gives the Nat. Verfed Sine of 36° }
17' his Summer Zenith Dist. at } .1939000
2 and 10

Winter Meridian Zenith Dist. 75° } .7414620

01' Nat. Verfed Sine }
Nat. Numb. found add .0764376

Gives the Nat. Verf. Sine of $10^{\circ} : 29'$ }
the Winter Alt. at 2 and 10 } .8178996

For the Altitude at 5 and 7, the Declination being the fame.

Fixed Number 9.7563844

Log. verf. Sine of 75° Time from Noon 9.8699242

Sum 9.6262086

Answering the Nat. Numb. .4228716

This

This Number added to the versed Sines of the Summer and Winter Meridian Zenith Distance above respectively, gives .5403340 and 1.1643336; wherefore the Summer Zenith Distance at 5 and 7, is $62^{\circ}:38'$, and because the other Number is greater than *Radius* (which informs us that the Sun is below the Horizion) the Winter Depression at at those Hours is $9^{\circ}:28'$. The Reader may compare this with the same Example wrought in *Hawney's Trigonometry* pag. 146.

C A S E IV.

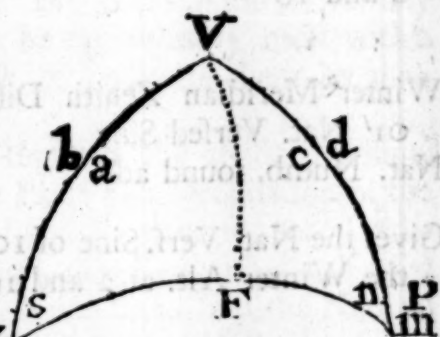
Given two Sides $OP = 72^{\circ}:00'$ and $OV = 52^{\circ}:00'$, with their contained Angle $VOP = 49^{\circ}:00'$; to find one of the other Angles, *P*.

ALGEBRAI-
CALLY:

By Theorem 12.

$$\frac{eb - afz}{as} = \frac{m}{n}. \text{ Or }$$

v being put =
versed Sine of the
given Ang. *O*, and
D = the Sine of *OZ*



the Diff. of *PO* and *OP*, by Theorem 13. $\frac{D + afv}{as}$

$$= \frac{m}{n}, \text{ or } \frac{D}{as} + \frac{fv}{s} = \frac{m}{n}.$$

Loga-

Logarithmic Operation, by Theorem 13.

$$OV = 520 : 00' \text{ L. s. } 9.8965321 \quad OP = 72^\circ : 00'$$

$$> O = 49 : 00 \text{ L. s. } 9.8777799 \quad OV = 52 : 00$$

$$\text{Sum} \quad 9.7743120 \quad \text{Diff. } 20 : 00 \text{ L. s. } 9.5340517$$

$$\text{Sum (just found) subt. } 9.7743120$$

$$\text{Diff. } 9.7597397$$

$$\text{Answ. Nat. Numb. } .5750947$$

$$\text{Ang. } O = 49^\circ : 00' \text{ L. ver. s. } 9.5364840$$

$$OP = 72 : 00 \text{ L. Cosine } 9.4899824$$

$$\text{Their Sum} \quad 9.0264664$$

$$\text{Ang. } O = 49 : 00 \text{ L. s. subt. } 9.8777799$$

$$\text{Diff. } 9.1486865 \text{ N. N. anf. } .1408272$$

$$\text{These two Nat. Numbers being added, give the Nat. } \left. \begin{array}{l} \text{Cotangent of Ang. } P = 54^\circ : 24' \end{array} \right\} .7159219$$

By letting fall a Perpendicular.

The Perpendicular VF being let fall from V upon OP ; in the right-angled Triangle OVF are known OV and the Ang. O , to find OF ; wherefore

By Theorem 3.

As Radius

10.

$$\text{To the Cosine of Ang. } VOP = \left. \begin{array}{l} 49^\circ : 00' \end{array} \right\} 9.8169429$$

$$\text{So is the Tangent of the Hypo-} \left. \begin{array}{l} \text{then use } OV = 52^\circ : 00 \end{array} \right\} 10.1071902$$

$$\text{To the Tangent of } OF = 40^\circ : 01 \quad 9.9241331$$

Now the Difference between OF and OP , i. e. $31^\circ : 59'$ is $= FP$; therefore from Theorem 25. inverted,

R

As

As the Sine of $OF = 40^\circ : 01'$	}	0.1917820
<i>Comp. Arith.</i>		
To the Cotangent of $Ang. O = 49^\circ : 00'$	}	9.9391631
So is the Sine of $FP = 31^\circ : 59$		
		9.7240075
To the Cotangent of $Ang. P = 54^\circ : 24'$, as before.	}	9.8549526

The Affection of the Arch lying between the given Angle and the Foot of the Perpendicular, may be known from what is said in Case 3. Var. 1. of right-angled Triangles; and comparing this Arch with the Length of the Side upon which the Perpendicular is designed to fall, it will be known whether the said Perpendicular fall *within* or *without* the Triangle; if it fall *within*, the Angle sought has the *same*, if *without*, a *different* Affection to the given Angle (*vide Princ. 18.*)

Also the Affection of the Arch lying between the Foot of the Perpendicular and the unknown Angle being determinable, and the Perpendicular being ever (by *Princ. 16.*) of the same Affection with the given Angle opposing it, from these (by *Princ. 17.*) the Affection of the third Side may be known, and so (though it happened to be omitted there) all the Ambiguities of Case 3. removed, in the Method of letting fall a Perpendicular.



CASE V,

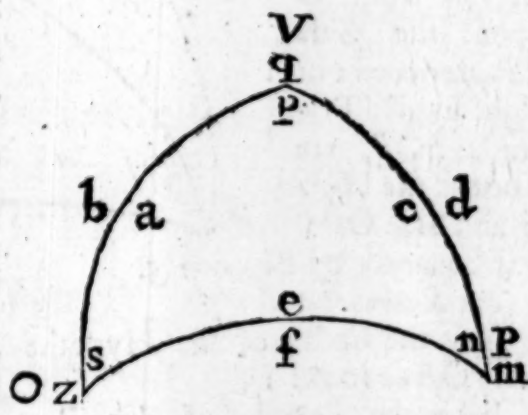
Given two Angles $P=80^{\circ}:12'$ and $O=86^{\circ}:11'$
with their interjacent Side $OP=104^{\circ}:15'$;
to find the other Angle V .

ALGEBRAI-
CALLY.

By Theorem 14.

$$\frac{zm + q}{sn} = f;$$

this is when
each of the
given Parts are
acute, but be-
cause OP is
obtuse f must



have a negative Sign, and so $\frac{zm + q}{sn} = -f$, and

by Reduction $zm + q = -f sn$, or $q = -f sn - zm$, i. e. $-q = f sn + zm$, which shews the Angle sought to be obtuse.

Logarithmic Operation of this Equation.

Ang. $O=86^{\circ}:11'$ L. s. 9.9990357 L. Cosine 8.8232404
Ang. $P=80^{\circ}:12'$ L. s. 9.9936160 L. Cosine 9.2309838

S. $OP=104^{\circ}:15'$ L. cs. 9.3912057 Sum 8 0542242

Sum 9.3838574 Anf. N. N°. .0113298

Anf. Nat. No.

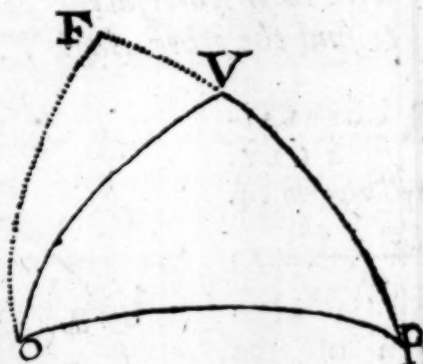
$$\begin{array}{r} .2420234 \\ .0113298 \\ \hline \end{array} = fns$$

$$\begin{array}{r} -.2533532 \\ \hline \end{array} = \text{the Nat. cs. of Ang. } V=104^{\circ}:41'$$

This Case may be solved in our Method by the Natural Canon, after the Manner that *Case 3.* was :
But it is by much the easiest solved by the Help of the versed Sines from Theorem 18.

By letting fall a Perpendicular.

The Perpendicular *OF* being let fall from *O* upon the Side *VP* produced ; in right-angled Triangle *FOP*, are known the *Ang. P* and Side *OP* ; consequently by *Theorem 5.*



As *Radius*

To the Cosine of the Hypothenuse $OP = 104^\circ : 15'$ } ^{10.} 9.3912057

So is the Tangent of *Ang. P* = } 10.7626322
 $80^\circ : 12'$

To the Cotangent of *FOP* = } 10.1538379
 $144^\circ : 57'$

Ang. OVP = $86 : 11$

Diff. $58 : 46$ = *Ang. FOV.*



2dly, By the Reverse of Theorem 26.

$$\text{As the Sine of Ang. } FOP = \} 0.2408679$$

144° : 57'. *Comp. Arith.*

$$\text{Is to the Cosine of Ang. } P = \} 9.2309838$$

80° : 12'

$$\text{So is the Sine of Ang. } FOV = \} 9.9319980$$

58° : 46'

$$\text{To the Cosine of Ang. } OVF = \} 9.4038497$$

75° : 19'

whose Supplement to 180° is 104° : 41' = Ang. *V* as before.

The Affection of the Angle (*FOP*) at the Perpendicular will always be known from what is subjoined to Var. 3. of *Case 3.* of right-angled Triangles ; now when this Angle (*FOP*) is *less* than the given Angle (*VOP*) adjoining to the Perpendicular, the Angle sought shall be of the *same* Kind with the given Angle (*VPO*) opposite to the Perpendicular, but when *greater*, of a *different*. Because in the former Case the Perpendicular (*FO*) will fall *within*, in the latter *without* the Triangle ; and therefore (by *Princ. 18.*) &c.



C A S E

CASE VI.

Given two Angles, $P = 80 : 12'$ and $O = 86 : 11'$, with their interjacent Side $OP = 104 : 15'$; to find one of the other Sides VP .

ALGEBRAICALLY. By Theorem 16. $\frac{d}{c} = \frac{fms + nz}{se}$ or if V be put for the verified Sine of OP , then $1 - v = f$, and $\frac{d}{c} = \frac{ms - msv + nz}{se} = \frac{S - msv}{se}$. See Theorem 17.

By letting fall a Perpendicular.

Drawing PF perpendicular to OV , or its Continuation; in the right angled Triangle OPF are known Ang. O , and the Hypothenufe OP ; therefore by Theorem 5.



As

As Radius
 To the Cosine of the Hypothe- } 10.
 nuse $OP = 104^\circ : 15'$ } 9.3912057
 So is the Tangent of $Ang. O = 86^\circ : 11'$ } 11.1757954

To the Cotangent of $Ang. OPF = 164^\circ : 50'$ } 10.5670011

$Ang. OPV = 80 : 12$

Diff. $84 : 38 = Ang. VPF.$

Then by Theorem 23. inverted.

As the Cosine of $Ang. OPF = 164^\circ : 50'$. *Comp. Arith.* } 0.0153967
 Is to the Cotangent of $OP = 104^\circ : 15'$ } 9.4047784
 So is the Cosine of $Ang. VPF = 14^\circ : 38'$ } 8.9709468

To the Cotangent of the Side $VP = 88^\circ : 35'$ } 8.3911219

The Angle OPF at the Perpendicular will ever (by Cor. 2. Prin. 7.) be *acute* or *obtuse*, as the given Side (OP) and the given Angle (VOP) opposite to the Perpendicular are of the *same* or of a *different* Affection; and, whether the Perpendicular fall within or without the Triangle, the Side sought shall be *acute* or *obtuse*, according as the given Angle opposite to the Perpendicular, and the Difference of the two Angles at the Perpendicular (to wit $Ang. VPF$) are of *like* or *unlike* Affections.

C A S E

C A S E VII.

Given two Sides $OP = 72^\circ : 00'$ and $OV = 52^\circ : 00'$ with an Angle $P = 54^\circ : 24'$ opposite to one of them; to find the Angle VOP contained between them.

ALGEBRAICALLY. Put p and q for the Sine and Cosine of the Angle

OPV then by Theorem 14. $\frac{mz + q}{sn} = f$ and

$$\frac{qz + m}{sp} = b \therefore mz + q = snf \text{ and } qz + m = bsp \therefore$$

$m^2z + qm = sfnm$ and $q^2z + qm = bspq$, and by exterminating qm we have $m^2z - q^2z = sfnm -$

$$bspq \therefore \frac{z}{s} = \frac{fnm - bpq}{m^2 - q^2}, \text{ or which is the same } \frac{z}{s} =$$

$$\frac{fnm - bq}{p^2 - n^2}, \text{ or } \frac{2nmf - 2pqb}{2p^2 - 2n^2}. \text{ Put } 2nm = P, \text{ which is}$$

the right Sine of an Arch double to that of the Angle OPV , whose versed Sine $= 2n^2$ call v ; and put $2pq$, which is the right Sine of an Arch, double to that of the Angle $OVP = V$, and its versed

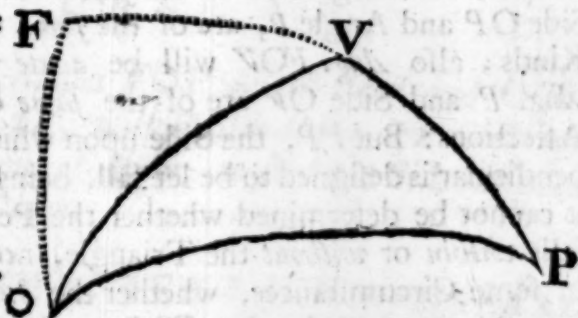
Sine $(= 2p^2) = v$, then will $\frac{z}{s} = \frac{Pf - Vb}{v - v}$ the Cotangent of the Angle POV .

Note, As $a : n :: e : p$, which is the Sine of the Angle OVP .

By

By letting fall a Perpendicular.

Letting the Perpendicular OF fall from O upon VP or its Prolongation, in the Triangle FOP , right-angled at F , are given *Ang. P* and Side OP ; consequently by *Theorem 5*.



As *Radius*

10.

To the Cosine of the Hypothuse $OP = 72^\circ : 00'$ } 2.4899824
 So is the Tangent of *Ang. P* = } 10.1451266
 $54^\circ : 24'$

To the Cotangent of *Ang. FOP* } 9.6351120
 $= 66^\circ : 39'$

Again, by *Theorem 23*.

As the Cotangent of $OP = 72^\circ : 00'$ } 0.4882240
Comp. Aritb. }
 Is to the Cosine of *Ang. FOP* = } 9.5989754
 $66 : 39$ }
 So is the Cotangent of $OV = 52 : 00$ } 9.8928098

To the Cosine of *Ang. FOV* = } 9.9791092
 $17 : 38$

Ang. FOP = $66 : 39$

Diff. $49 : 01$ = the required *Ang. VOP*.

or rather $49^\circ : 00'$ (*vide* Case 4.)

S

In

In the Solution of this Case the Affection of the Angle (*FOP*) at the Perpendicular will always be known, as being *acute* or *obtuse*, according as the Side *OP* and Angle *P*, are of the *same* or *different* Kinds; also *Ang. FOV* will be *acute* or *obtuse* as *Ang. P* and Side *OV* are of the *same* or *different* Affections: But *VP*, the Side upon which the Perpendicular is designed to be let fall, being unknown, it cannot be determined whether the Perpendicular falls *within* or *without* the Triangle, nor therefore, in some Circumstances, whether the *Sum* or *Difference* of the two Angles *FOP* or *FOV* constitute the required Angle *VOP*; but observe, if the *Ang. FOV* be less than *FOP*, and the Sum of these two Angles (*FOV* and *FOP*) less than two right Angles, then *Ang. FOV* either added to, or subtracted from *FOP*, will give the Angle *VOP*; which therefore is ambiguous. But if *FOV* be not less than *FOP*, or the Sum of *FOV* and *FOP* not less than a Semicircle; then their Sum in the former, and their Difference in the latter Circumstances, shall give the single Value of the *Ang. VOP*, which then is not ambiguous.

It appears then, that this Example of ours is ambiguous, and so that the Angle *VOP* may be accounted $84^{\circ} : 17'$ as truly as $49^{\circ} : 00'$; and from hence we may infer, that tho' *Case 4.* be not ambiguous, yet there are two Angles (*to wit* $49^{\circ} : 00'$ and $84^{\circ} : 17'$) either of which used along with the same containing Sides ($52^{\circ} : 00'$ and $72^{\circ} : 00'$) will give the required Angle (*P*) of the same Number ($54^{\circ} : 24'$) of Degrees: And so of other Examples of that kind.

C A S E VIII.

Given two Angles $VOP = 49^\circ : 00'$ and $OPV = 54^\circ : 24'$, with a Side $OV = 52^\circ : 00'$ opposite to one of them; to find the Side included between them.

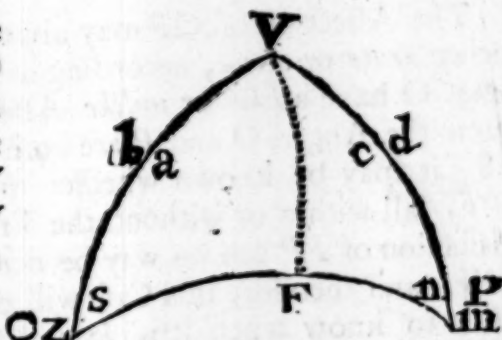
A L G E B R A I C A L L Y: By Theorem 22.
 $\frac{f}{e} = \frac{abz - cdm}{a^2 - c^2}$, or $\frac{2abz - 2cdm}{2a^2 - 2c^2}$. Put the Sine of twice $OV (= 2ab) = A$, its verfed Sine $(= 2a^2) = v$. The Sine of twice $VP (= 2cd) = B$, and the verfed Sine $(= 2c^2)$ call v , then will $\frac{f}{e} = \frac{Az - Bm}{v - v}$ the Cotangent of the Side OP .

Note, To find VP , say as $n : a :: s : c$ the Sine of VP .

By letting fall a Perpendicular.

Here because the Angles O and P are both of a Kind, the Perpendicular VF , let fall from V (by *Prin.* 18.) falls within the Triangle. Now OZ

in the Triangle VOF are given VO and *Ang.* O ; wherefore by Theorem 3.



S 2

As

As Radius		10.
Is to the Cosine of <i>Ang. VOP</i> =	}	9.8169429
49° : 00'		
So is the Tangent of the Hypo-	}	10.1071902
then use <i>OV</i> = 52° : 00'		
To the Tangent of <i>OF</i> = 40° : 01'		<u>9.9241331</u>

Again, by Theorem 25.

As the Cotang. of <i>Ang. VOP</i> =	}	0.0608369
49° : 00' <i>Comp. Arith.</i>		
Is to the Sine of <i>OF</i> =	}	9.8082180
40 : 01		
So is the Cotangent of <i>Ang. VPO</i>	}	9.8548704
= 54 : 24		
To the Sine of <i>FP</i> =	}	9.7239253
31 : 59		

OF = 40 : 01

Sum 72 : 00 = *OP* sought,

The Affection of *OF* may always be known, as being *acute* or *obtuse*, according as the Side *OV* and *Ang. O* have a *like* or *unlike* Affection; also because the Angles *O* and *P* are both given (by *Prin.* 18.) it may be known whether the Perpendicular (*VF*) fall within or without the Triangle; but the Affection of *FP* can no way be determined universally, and therefore this Case will often be ambiguous; to know when it is so and when not, take these Rules,

I. When the Perpendicular falls within the Triangle, that is, when the two given Angles have the same Affection.

To

To the acute Value of FP , and also to its *obtuse* one, add OF ; and if each of these Sums is less than a Semicircle, then either the acute Value of FP , or its *obtuse* one added to OF , gives the Value of OP , which then is ambiguous. But when only one of these Sums is less than a Semicircle, the acute Value of FP added to OF gives only the Value of OP , which then is not ambiguous.

2. When the Perpendicular falls without the Triangle, *i. e.* when the two given Angles are of dissimilar Affections.

When the *obtuse* Value of FP is less than OF , OP will be had by subtracting either Value of FP from OF ; and then (OP) is ambiguous: But when the *obtuse* Value of FP is not less than OF , the *acute* Value of FP taken from OF leaves the only Value OP ; which therefore cannot then be ambiguous.

CASE IX.

Given two Sides $OV = 71^\circ : 50'$ and $OP = 119^\circ : 54'$, with an Angle $OVP = 145^\circ : 10'$ opposite to one of them; to find the third Side VP .

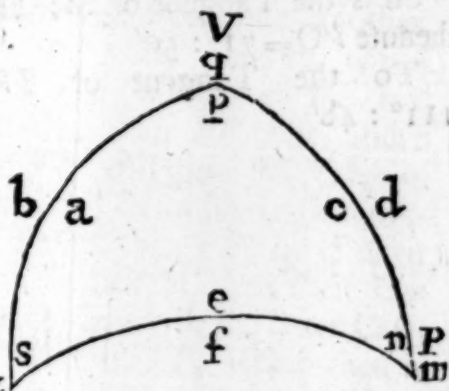
ALGEBRAICALLY.

By Theorem 20.

$$\frac{d}{c} = \frac{efm - abq}{e^2 - a^2} \text{ or}$$

$$\frac{efm - abq}{b^2 - f^2}; \text{ but}$$

since f and q in this Example are



$$\text{put for the Cosines of obtuse Arches } \frac{d}{c} = \frac{abq - efm}{b^2 - f^2}$$

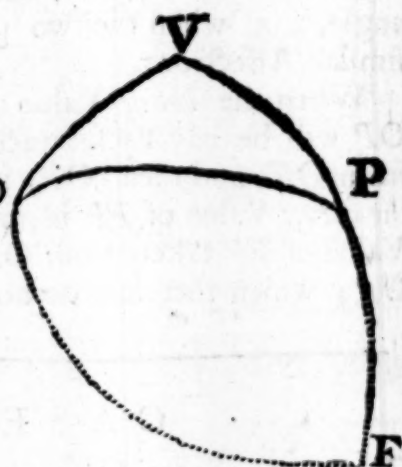
Now if A and v be put for the right and versed Sine

Sine respectively of that Arch which is double the Number of Degrees to that which b is the Sine of, B and v the right and ver. Sine of that Arch which is double to that which f is the Sine of, then we

have $\frac{d}{c} = \frac{Aq - Bm}{v - v}$ the Cotangent of VP .

By letting fall a Perpendicular.

The Perpendicular OF being drawn to PV produced, in the Triangle FVO , right angled at F , are known the Hypotenuse $OV = 71^\circ : 50'$, and the Angle $OVF = 145^\circ : 10'$; hence by Theorem 3.



As Radius		10.
To the Cosine of Ang. $FVO = 145^\circ : 10'$	$\left. \begin{array}{l} \\ \end{array} \right\}$	9.9142464
So is the Tangent of the Hypotenuse $VO = 71^\circ : 50'$	$\left. \begin{array}{l} \\ \end{array} \right\}$	10.4839425
To the Tangent of $FV = 111^\circ : 48'$	$\left. \begin{array}{l} \\ \end{array} \right\}$	10.3981889

Again

Again, by Theorem 24.

As the Cosine of $OV =$	$\}$	0.5061487
$71^{\circ} : 50'$ Comp. Ar.	$\}$	
Is to the Cosine of $FV =$	$\}$	9.5698043
$111 : 48$	$\}$	
So is the Cosine of $OP =$	$\}$	9.6976545
$119 : 54$	$\}$	
To the Cosine of $FP =$	$\}$	9.7736075
$53 : 35$	$\}$	
$FV =$ $111 : 48$		
Diff. $58 : 13 = VP$ required.		

In this Case the Affection of FV will be known, as being *acute* or *obtuse*, according as OV and the Angle OVP have a *like* or *unlike* Affection, as also that of FP , it being *acute* or *obtuse*, as the Angle OVP and Side OP are of a similar or dissimilar Kind; but it can't universally be known whether the Perpendicular OF falls within the Triangle or without, upon VP prolonged, and so whether VP and PF added together, or subtracted from one another, give the required Side VP : But observe, that if PF be less than VF , and also the Sum of VF and PF less than a Semicircle; then PF either added to, or subtracted from VF , will give the Value of VP , which therefore in such Circumstances is ambiguous. But if FP be not less than VF , or if their Sum be not less than a Semicircle, then their *Sum* in the former, and their *Difference* in the latter Case, shall give one single Value of VP , which then is therefore not ambiguous.

C A S E

CASE X.

Given two Angles, $VOP = 49^\circ : 00'$ and $VPO = 54^\circ : 24'$ with a Side, $OV = 52^\circ : 00'$ opposite to one of them; to find the third Angle OVP .

ALGEBRAICALLY. By Theorem 19. $\frac{q}{p} = \frac{szb - nmd}{n^2 - s^2}$ where d is the Cosine of VP , or if A and v be put for the right and ver. Sine of an Arch, which is double to that which s is the Sine of, B and v the right and ver. Sine of an Arch that is double to that which n is the Sine of, then $\frac{q}{p} = \frac{Ab - Bd}{v - v}$ the Cotangent of the Angle OVP .

* By letting fall a Perpendicular.

Because the two given Angles O and P have a like Affection, a Perpendicular (VF) let fall from V upon OP , will (by *Princ.* 18.) fall within the Triangle: now in the Triangle OVF , right-angled at F , are known OV and Angle O ; wherefore by *Theorem 5*.

As Radius

To the Cosine of the Hypothenufe

$VO = 52^\circ : 00'$

So is the Tangent of Ang. VOP

$= 49^\circ : 00'$

To the Cotangent of Ang. OVF

$= 54^\circ : 42'$

10.

9.7863420

10.0608369

9.8501789

* See Fig. p. 131.

Again,

Again, by Theorem 24.

$$\text{As the Cosine of Ang. O} = \left. \begin{array}{l} 49^\circ : 00' \text{ Comp. Arith.} \end{array} \right\} 0.1830571$$

$$\text{Is to the Sine of Ang. OVF} = \left. \begin{array}{l} 54 : 42 \end{array} \right\} 9.9117634$$

$$\text{So is the Cosine of Ang. P} = \left. \begin{array}{l} 54 : 24 \end{array} \right\} 9.7650147$$

$$\text{To the Cosine of Ang. PVF} = \left. \begin{array}{l} 46 : 24 \end{array} \right\} 9.8598352$$

$$\text{Ang. OVF} = 54 : 42$$

Sum 101 : 06 = the Ang. OVP required.

In this Case it will always be known from the *Similarity* or *Dissimilarity* of the two given Angles O and P, whether the Perpendicular VF, falls *within* or *without* the Triangle; also the Affection of Ang. OVF will be known, as being *acute* when OV and Ang. O have the *same*, and *obtuse* when they have a *different* Affection; but the Affection of Ang. FVP cannot be universally known, which Defect will sometimes cause this Case to be ambiguous; when it does, and when not, may be known by these Rules:

1. When the given Angles are of the same Kind; to the acute Value of Ang. FVP, and also to its obtuse one, add the Angle OVF; if each of these Sums be less than two right Angles, then either the acute Value of Ang. FVP, or its obtuse one added to Ang. OVF, gives a Value of Ang. OVP; which therefore is then ambiguous: But when only one of these Sums is less than a Semicircle, the acute Value of Ang. FVP, added to Ang. OVF, gives the only Value of Ang. OVP; which then is not ambiguous.

T

1. When

2. When the given Angles are of different Species; when the obtuse Value of *Ang. FVP* is less than the *Ang. OVF*, the *Ang. OVP* may be had by subtracting either Value of *Ang. FVP* from *Ang. OVF*, and is then ambiguous: But when the obtuse Value of *Ang. FVP* is not less than *Ang. OVF*, the acute Value of *Ang. FVP*, taken from *Ang. OVF*, gives the single Value of *Ang. OVP*; which therefore in such Circumstances is not ambiguous.

CASE XI.

Given the three Sides $OV = 71^\circ : 50'$, $OP = 119^\circ : 54'$ and $VP = 58^\circ : 12'$ of a Spherical Triangle *VOP*; to find an Angle *OVP*.

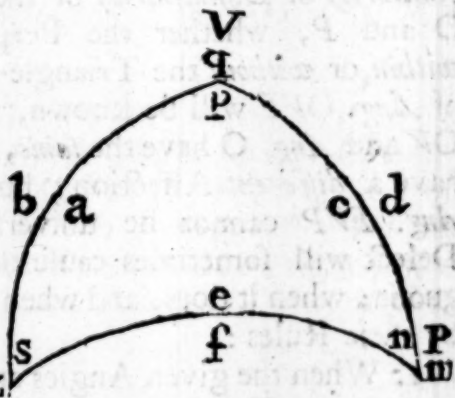
ALGEBRAI-
CALLY.

Because the Side *OP* is obtuse, and the other two acute, by *Theorem 8.* $acq + bd = -f$, ergo $acq = -f - bd$, and consequently

$q = \frac{-f - bd}{ac}$, which being negative, shews the re-

quired Angle to be obtuse: Or putting v = the versed Sine of the Angle *OVP*, K = the versed Sine of the Difference of the two containing Sides *OV* and *VP*, V = the versed Sine of the Side *OP*, and $\mathcal{Q}$ = the versed Sine of the Sum of *VO* and *VP*;

by *Theorem 9.* $acv + K = V$, and $v = \frac{V - K}{ac}$, or by
Theo-



Theorem 10. $\frac{1}{2} vQ - \frac{1}{2} vK + K = V$, and by Reduction, $v = \frac{V-K}{\frac{1}{2}Q - \frac{1}{2}K}$

Logarithmic Operation of the Equation $q = \frac{-f-bd}{ac}$;

i. e. Solution by the Inverse of Theorem 8.

L. cs. of $OV = 71^\circ : 50'$ 9.4938513

L. c. s. of $VP = 58 : 12$ 9.7217742

	Sum	9.2156255	
Anf. Nat. Numb.		.1642954 = bd	
	Nat. cs. of $OP = 119^\circ : 54'$	.4984877 = f	
	Nat. No. found = bd , add	.1642954	
L. s. of $OV = 71^\circ : 50'$	9.9777938		
L. s. of $VP = 58 : 12$	9.9293641	Sum = $-f-bd$	.6627831
	Sum	9.9071579	L. of $-f-bd$ 9.8213714
			Subt. 9.9071579

Remains the L. cs. of the Ang. $V = 145^\circ : 10'$ 9.9142135

But the Equation $q = \frac{-f-bd}{ac}$ may be expressed

thus ; $q = \frac{-f}{ac} - \frac{b}{a} \frac{d}{c}$; wherein $\frac{b}{a} \frac{d}{c}$ is the Rectangle of the Cotangents of the Sides VO and VP , and so the Operation may be reduced to a Form a small Matter more simple, thus ;

L. s. of $OV = 71^\circ : 50'$ 9.9777938 L. cs. of OP } 9.6976545
 L. s. of $VP = 58 : 12$ 9.9293641 = $119^\circ : 54'$ }

Sum	9.9071579	Subt.	9.9071579
L. ct. of $OV = 71^\circ : 50'$	9.5160575	Remains	9.7904966
L. ct. of $VP = 58 : 12$	9.7924101	Anf. N. No	.6173004

Sum	9.3084676	
Anf. Nat. Numb.	.2934547	Add .2034547

Sum the N. cs. of Ang. $P = 145^\circ : 10'$.8207551

Operation from the same Theorem, by the Natural Canon.

	9	1		
VO=	71	: 50		
VP=	58	: 12		
Sum	130	: 02	N. cs.	.6432332
Diff.	13	: 38	N. cs.	.9718240
			Sum	1.6150572
			Half Sum	.8075286 = (per Rule 2. p. 79) to ac, the Rec. of the Ss. } of VP & VO
			Diff.	.3285908
			Half Diff.	.1642954 = bd = the Rect. of the cs.
Nat. cs. of OP=	119°	: 54'		.4984877
Rect. of the Sines of VO and VP, add				.1642954

Sum .6627831 which divided by (.8075286) the Rectangle of the Sines of VO and VP, gives .8207549 = the Nat. Cofine of the Angle OVP, to which answers 145° : 10', as before.

Logarithmic Operation by the Inverse of Theorem 9.

Natural versed Sine of OP =			1.4984877
Side OV =	71°	: 50'	
Side VP =	58	: 12	
Diff.	13	: 38	Nat. versed Sine .0281760
			Difference 1.4703117
OV=71° : 50' L. s.	9.9777938	Logarithm	0.1674094
VP=58 : 12 L. s.	9.9293641		
Sum	9.9071579	Subt.	9.9071579
Remains the Log. versed Sine of Angle QVP =			0.2602515
145° : 10', as before.			Loga-

Logarithmic Operation according to Theorem 11.

$$OP = 119 : 54 \quad \text{Nat. versed Sine } 1.4984877$$

$$OV = 71 : 50$$

$$VP = 58 : 12$$

$$\text{Diff. } 13 : 38 \quad \text{Nat. versed Sine } .0281760$$

$$\text{Difference } 1.4703117 \quad \text{Log. } 0.1674094$$

$$\text{Sum } 130 : 02 \quad \text{Nat. versed Sine } 1.6432332$$

$$\text{Diff. } 13 : 38 \quad \text{Nat. versed Sine } .0281760$$

$$\text{Difference } 1.6150572$$

$$\text{Half Diff. } .8075286 \quad \text{Log. } 9.9071579$$

$$\text{Diff. equal to the Log. versed Sine of Ang. } V = \left. \begin{array}{l} 145^{\circ} : 10' \\ \end{array} \right\} 0.2602515$$

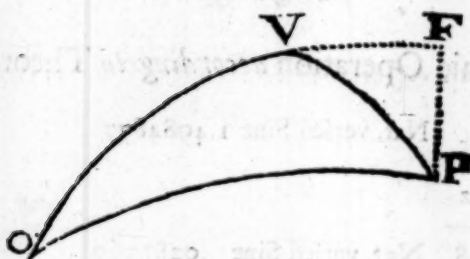
To solve this Case by letting fall a Perpendicular.

Call one of the Sides comprehending the enquired Angle the true Base ; then as the Tangent of half the true Base : is to the Tangent of Half the Sum of the other two Sides : : so is the Tangent of Half the Difference of those two Sides : to the Tangent of Half the alternate Base.

If the *true* exceeds the *alternate* Base, then the Perpendicular falls upon the true Base *within* the Triangle ; but if the *alternate* exceeds the *true* Base *without* the Triangle, upon the true Base prolonged.

Whether the Perpendicular fall within or without the Triangle, the *Sum* of the true and alternate Semi-bases, gives the Base of the *greater* of the two right-angled Triangles into which the oblique One is divided, and their *Difference* that of the *lesser* ; whence the required Angle may be found by one of the Cases of right-angled Triangles.

Ope-



Operation, calling *VO* the true Base

True Base <i>VO</i> =	$71^{\circ} : 50'$	Side <i>OP</i> =	$119^{\circ} : 54'$
Half	$35 : 55$	Side <i>VP</i> =	$58 : 12$
		Sum	$178 : 06$
		Half Sum	$89 : 03$
		Diff.	$61 : 42$
		Half Diff.	$30 : 51$

As the Tangent of half the true Base = $35^{\circ} : 55'$ } *Comp. Arith.* } 0.1400679
 Is to the Tangent of half the Sum of the Sides } 11.7803592
 89° : 03' }
 So is the Tang. of half the Diff. of the Sides = $38^{\circ} : 51'$ } 9.7761947

To the Tang. of half the Alternate Base = $88^{\circ} : 51'$ } 11.6966218

True Semi-Base	$25^{\circ} : 55'$	add and subtract.
Sum = Base <i>OF</i> =	$124 : 46$	
Diff. = <i>VT</i>	$52 : 56$	

In this Example the *Alternate* is found to exceed the *true* Base, and consequently Perpendicular *PF* is thereby known to fall without the Triangle; wherefore in the right-angled Triangle *VPF* are known *VP* and *VF*; consequently by *Theorem* 3. inverted.

As

As the Tangent of $VP = 58^\circ : 12'$ 10.2075899
 To the Tangent of $VF = 52 : 56$ 10.1218346
 So is Radius 10.

To the Cosine of $Ang. FVP = 34^\circ : 50'$ 9.91142447

But the Angle FVP is the Supplement of the required Angle OVP to a Semicircle, and consequently 180° Minus $34^\circ : 50' = 145^\circ : 10'$ is the Angle OVP , as before.

CASE XII.

Given three Angles $O = 86^\circ : 11'$, $V = 104^\circ 41'$, and $P = 80^\circ : 12'$, of a Spherical Triangle OVP , to find the Side, as OP .

ALGEBRAICALLY: By Theorem 14.
 Since the Angle V is obtuse, $\frac{zm-q}{sn} = f$.

* Logarithmic Operation by Theorem 14.

$O = 86^\circ : 11'$ L. c. s. 8.8232404
 $P = 80 : 12$ L. c. s. 9.2309838 N. c. s. Ang. V is — .2554766

Sum 8.0542242 Nat. No. answer. + .0113298

$O = 86^\circ : 11'$ L. s. 9.9990357 Diff. — .2421468

$P = 80 : 12$ L. s. 9.9939160 Logarith. 9.3840787

Sum 9.9926517 Subtract 9.9926517

Remains the Log. Cosine of OP (obtuse because zm is less than q , and so the Ballance Negative) = $104^\circ : 15'$ 9.3914270

* See Fig. p. 131.

Part

Part of this Process might have been perform'd by Cotangents, as the second Operation of Case 11. was.

Operation according to the same Theorem, by the Natural Canon.

Ang. O = 86 : 11
Ang. P = 80 : 12

Sum 166 : 23 Nat. Cofine .9718925

Diff. 5 : 59 Nat. Cofine .9945522

Sum 1.9664447

Half Sum .9832223 = *sn* (per Rule 2. P. 79.)

Difference .6226597

Half Diff. .6113298 = *zm* (per the same Rule.)

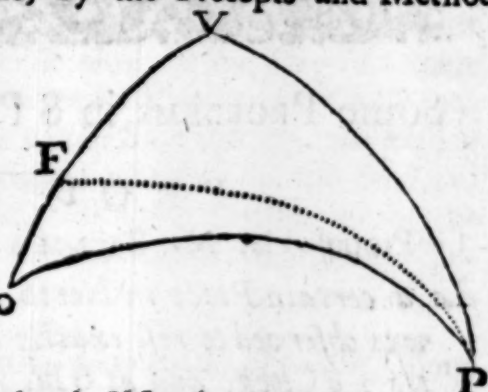
Angle V = 104° : 41' Nat. Cofine - .2534766 = *q*
Rect. of the Cofines of O and P + .0113298 = *zm*

Difference - .2421468, which divided by *sn* (=the Rect. of the Sines of O and P = .9832223) quotes - .2462788 = the Nat. Cofine of OP 104° : 15', as before.

To solve this Case by letting fall a Perpendicular.

Take the Supplement of one of the Angles adjacent to the Side required for one, and the other two Angles for the other two Sides of another Triangle; the Triangle thus composed shall have its Angles equal to the Sides of that given; and having thus changed the Angles into Sides, the required Side may be sought for as an Angle opposite to the Side produced from the Angle which originally opposed the

the enquired Side, by the Precepts and Method used in the last Case. Thus taking the Supplement of Angle O (one of the Angles adjacent to the Side OP required) for the Side VP; the Angle P for the Side OV; and the Angle V for the Side OP; in the Triangle OVP are known all the Sides, to find the Angle OVP, equal to the required Side OP in the original Triangle: Wherefore making OV the true Base,



As the Tang. of half the true Base = 40 : 06 C. Ar. 0.0746476
 Is to the Tang. of half the Sum of the Sides = } 10 7881847
 99° : 15'

So is the Tang. of half their Diff. = 5 : 26 8 9782483

To the Tangent of half the alternate Base = } 9.8410806
 34 : 45

True half Base = 40 : 06

Sum = VF = 74 : 51

Diff. = FO = 5 : 21

Again, by Theorem 3. inverted,

As the Tangent of VP = 93° : 49' 11.1757954

To the Tangent of VF = 74° : 51' 10.5674201

So is the Radius 10.

To the Cosine of the Angle OVP = } 9.3916247
 104° : 15'

So the required Side PO is 104° : 15', as found
 U Some

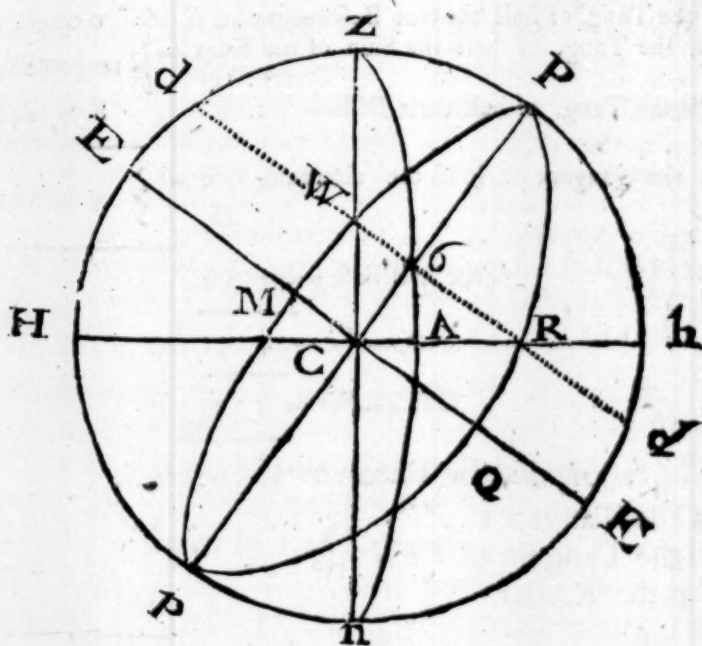


Some PROBLEMS in SPHERICS.

PROB. I.

Proposed by Mr. BULMAN of Deptford.

In a certain Place in North Latitude, the Sun was observed to rise exactly at 58 Minutes after 3 o'Clock, and at 6 his Altitude was taken, (the same Morning) and found to be $15^{\circ}:20'$, his Declination being North. Requir'd the Latitude of the Place where, and the Day of the Year when these Observations were made?



SOLUTION.

LET the Circle $HEZP_{bn}$ represent the Meridian of the Place of Observation, Hb the Horizon

Horizon, Zn the Prime Vertical, EE the Equator, Pp the Axis of the World, and dd the Parallel of the Sun's Declination on the Day of Observation; then will 6 be its Place at 6 o'Clock and R at Rising: Put L = the Sine and l = the Cosine of the required Latitude; as also D = the Sine and d = the Cosine of the unknown Declination of the Sun; then 1st, in the Rectangular Spherical Triangle CRQ , by *Trig. Theorem 2 inverted*, as

$\frac{L}{l}$ (= the Tangent of the Angle RCQ = the Comp.

of the Lat.) : is to $\frac{D}{d}$ (= the Tangent of RQ the the Sun's Declination) : : so is 1 (= Rad.) : to

$\frac{LD}{ld}$ = the Sine of CQ the ascensional Diff. (which is given = 6 Hours — 3 Hours — 58 Minutes = 2 Hours : 2' = $30^\circ : 30'$, whose Sine call m .) Ergo

$\frac{LD}{ld} = m$. Again, 2^{dly}, In the Spherical Tri-

angle $C6A$, Right Angled at A , by *Trig. Theorem 1*. as 1 (= Rad.) : to D (= the Sine of $C6$ = the Declination of the Sun) : : so is L (= the Sine of the Angle $6CA$ = the Lat. of the Place) : to LD = the Sine of $6A$ = the Altitude of the Sun at Six; but this Altitude was given $15^\circ : 20'$, the Sine of which call Q ; then will LD be = Q ; substitute

Q for LD in the Equation $\frac{LD}{ld} = m$, and we have

$\frac{Q}{ld} = m$; hence $ld = \frac{Q}{m}$; wherefore Q is the Rec-

tangle of the Sines, and $\frac{Q}{m}$ that of the Cosines of the Sun's Declination and the Place's Latitude; consequently, since both these Arches are known to be

U 2

acute

acute, by *Theorem 2* and 4. p. 3. $\frac{Q}{m} - Q$ will be equal to the Cosine of the Sum, and $\frac{Q}{m} + Q$ equal to the Cosine of the Difference of the Sun's Declination and Place's Latitude. Whence this universal Rule for all Questions of this Nature.

Divide the Sine of the Sun's Altitude at six, by the Sine of his Ascensional Difference; to this Quotient add the Sine of the Altitude at Six; also take the Difference between this Quotient and the Sine of the Sun's Altitude at Six; the Difference is the Cosine of the Sum, and the Sum the Cosine of the Difference of the Place's Latitude, and Sun's Declination.

Find in the Tables the Arches answering to these two Cosines, half the Sum of the answering Arches is the Latitude of the Place, and half their Difference the Declination of the Sun.

Hence this Observation was made in Latitude North $56^{\circ} : 41' : 7''$, and at a Time when the Sun had $18^{\circ} : 26' : 51''$ of North Declination, that is on May 2.

SCHOLIUM.

Since above DL is found equal Q , and $\frac{L D}{l d} = m$;

the Question is reduced to this Problem, purely Geometrical, viz. Given the Rectangle of the Sines of two Arches, as also the Rectangle of their Tangents, to find the Arches themselves; wherefore the above Rule, by changing every where the Words *Sine of the Sun's Altitude at Six* into *Rectangle of the Sines*, and *Sine of his Ascensional Difference* into *Rectangle of the Tangents*, becomes a Theorem for finding those Arches, as well as the Latitude and Declination required in this Question. By a different Process the following Rule is gain'd, viz.

As

As the Sine of the Sun's Ascensional Difference : is to the Sine of his Altitude at Six, :: so is Radius : to a fourth Number, which being taken from the covered Sine of the Sun's Altitude at Six, leaves the versed Sine of the Difference between the Sun's Declination and Place's Latitude, and being added to twice the Sine of the Sun's Altitude at Six, gives the versed Sine of the Sum of the Latitude and Declination of the Sun.

N. B. When the greater Arch found by either of these Rules exceeds ($23^{\circ} : 29'$) the Obliquity of the Ecliptic, it always represents the Place's Latitude, as the lesser Arch does the Declination of the Sun ; so that then the Question can admit of but one Answer ; but within the Tropics, *i. e.* when the greater Arch found by this Rule is less than $23^{\circ} : 29'$, the greater or lesser Arch either may represent the Latitude or the Sun's Declination ; and so Questions of this kind will, in that Case, admit of two Answers.



PROB.

PROB. II.

In a certain Northern Latitude, on board a Ship at Sea, the Sun's Altitude when due West was observed to be $29^{\circ} : 32'$, and afterwards, the same Day at Six o'Clock, his Altitude was found to be $18^{\circ} : 49'$; required the Latitude of the Place where, and Declination of the Sun when these Observations were made?

SOLUTION.

IN the Figure belonging to the last Problem, let *W* represent the Place of the Sun when due West, and *6*, as there, the Place thereof at Six o'Clock; also put *b* = the Nat. Sine of the given Alt. of the Sun at Six = $18^{\circ} : 49'$; *d* = the Nat. Sine of his Alt. when due West = $29^{\circ} : 32'$, and *x* = the Nat. Sine of his unknown Declination; then (1st) in the Spherical Triangle *CMW*, right-angled at *M*, by *Trig. Theorem 1.* inverted, as *d* (= the Sine of the Hypotenuse *WC* = the Altitude of the Sun when due West) : is to **1** (= *Rad.*) :: so is *x* (= the Sine of *WM* = the Sun's Declination) : to $\frac{x}{b}$ = the Sine of the Angle *MCW* = the Latitude of the Place. Again, (2^{dly}) in the Triangle *C6A*, right-angled at *A*, as *x* (= the Sine of *C6* = the Declination of the Sun) : is to **1** (= *Rad.*) :: so is *b* (= the Sine of *6A*, the Sun's Alt. at Six) : to $\frac{b}{x}$ (= the Sine of the Angle *6CA*, the Latitude of the Place); Ergo $\frac{x}{d} = \frac{b}{x}$, and $xx = db$, which multiply'd

triply'd by 2, gives $2xx = 2db$; but $2xx$ is known by a common Theorem, to be equal to the versed Sine of twice the Arch whose Sine is x , and by Rule 1st and 2d p. 79, $2bd$ is equal to the Difference or Sum, according as the Sum of the two Arches is acute or obtuse, of the Cosines of the Sum and Difference of the two given Altitudes; whence we have gain'd for solving all Questions of this Sort, this universal Theorem.

Take the Sum and Difference of the given Altitudes of the Sun when due West, and at Six o'Clock, and also their natural Cosines; if the Sum be obtuse, add together the Cosines of the Sum and Difference, but if acute take their Difference, which Sum or Difference, as the Case requires, shall be the natural versed Sine of an Arch double to the Declination of the Sun; whence the Latitude may be had by a common Case in Spherical Trigonometry, to wit, by one of the Analogies used in the Process of this Solution.

Hence the Latitude is found $54^{\circ} : 00'$, and the Declination of the Sun $23^{\circ} : 29'$ North.

It is needless to inform the Reader, that this Theorem will serve equally well for finding the Latitude of the Place and Declination of the Sun, from an Observation of his Altitude when due East, and at 6 in the Morning.

PROB.

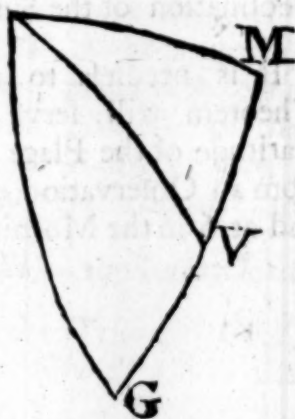
P R O B. III.

Being the Prize Question in the *Ladies Diary* 1739.

Given the Latitudes of three Places, Moscow $55^{\circ} : 30'$, Vienna $48^{\circ} : 12'$, Gibraltar $35^{\circ} : 30'$, all lying directly in the same Arch of a great Circle; the Difference of Longitude between Vienna (situate in the Middle) and Moscow easterly, is equal to that between Vienna and Gibraltar westerly; it is required to find the true Bearing and Distance of each Place from the other, and the Difference of their Longitudes according to the Convexity of the Globe?

SOLUTION.

LET N represent the North Pole, M Moscow, V Vienna, and G Gibraltar; then will the Arches NM NV and NG represent the Complements of the Latitudes of these three Places respectively. Put a = the Sine, and b = the Cosine of the Lat. ($55^{\circ} : 30'$) of Moscow; c = the Sine and d the Cosine of the Lat. ($=48^{\circ} 12'$) of Vienna, and e = the Sine and f = equal the Cosine of the Latitude of Gibraltar ($=35^{\circ} 30'$); also x = the Sine and y = the Cosine of the Angle GNV = Ang. VNM = the common Diff. of Long. of the two extrem Places from that in the Middle. Then by



by Trig. Theorem 12. $\frac{xb}{ad-byc}$ will be equal to the

Tangent of the Angle NVM , and $\frac{xf}{fyc-ed}$ that of the Angle NVG ; but these Angles being supplemental to each other, their Tangents are equal;

consequently $\frac{xb}{ad-byc} = \frac{xf}{fyc-ed}$ and $\frac{b}{ad-byc} =$

$\frac{f}{fyc-ed}$; Ergo $\frac{af+be \times d}{2bf \times c}$; but $af+be$ (by Theo-

rem 1. p. 3.) is equal to the Sine of the Sum of the Arches NM and $NG = 89^\circ$, which call n , and

$\frac{d}{c}$ is the Cotangent of $(48^\circ:12)$ the Latitude of

Vienna, which put $= t$, then will $\frac{nt}{2bf}$ be $= y =$

.969343 the Cosine of $14^\circ:13':27'' =$ the Angle MNV or $GNV =$ the Difference of Longitude of either *Moscow* or *Gibraltar* from *Vienna*.

Hence it may be easily found; that *Vienna* bears from *Moscow* South $56^\circ:4'$ westerly; *Moscow* from *Vienna*, North $44^\circ:50'$ easterly; *Gibraltar* from *Vienna*, South $44^\circ:50'$ westerly; *Vienna* and *Moscow* from *Gibraltar*, North $35^\circ:16'$ easterly.

Vienna is distant from *Gibraltar* $16^\circ:29' = 1146$ Miles; *Vienna* from *Moscow* $11^\circ:23' = 791$ Miles; *Gibraltar* from *Moscow* $27^\circ:52' = 1937$ Miles. Q. E. I.

X

PROB

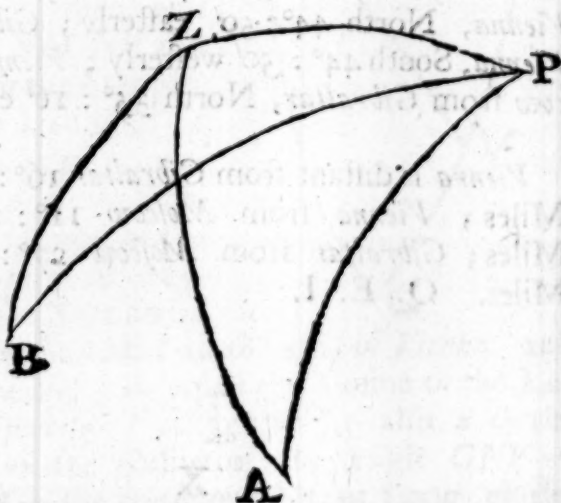
P R O B. IV.

Being the 2d Question in the *Ladies Diary* 1740.
Proposed by Mr. NICH. FARRER.

Sometime in the Spring Quarter in 1739, in the Forenoon, an Observation being made upon the Sun, his Altitude was found $33^{\circ}:41':40''$, and Azimuth from the North $102^{\circ}:40':52''$; and some time after, on the same Forenoon, his Altitude was found $48^{\circ}:46':53''$, and Azimuth $134^{\circ}:39':56''$. From whence the Latitude of the Place of Observation, Month, Day, and Hours of Observations may be found, and are here requir'd; with a general Theorem for all Questions of this Nature.

SOLUTION.

LET
Pre-
sent
the North
Pole; Z
the Zenith
of the Place
of Obser-
vation; A
the Place
of the Sun
at the first,
and B at
the second



Observation; then will ZP be equal to the Com-
pliment

pliment of the unknown Latitude, ZA the Complement of the Sun's Altitude at the 1st Observation. ZB at the second; the Angle PZA the Azimuth of the Sun from the North at the first, and $Ang. PZB$ at the second Observation; and the Arches PA and PB the Compliments of the Sun's Declination at the first and second Observations. Put $A =$ the Sine and $a =$ the Cosine of $(33^\circ:41':40'')$ $H =$ the Sine, and $b =$ the Cosine of $(48^\circ:46':53'')$ $m =$ the Cosine of $(162^\circ:40':52'')$ the lesser Azimuth, $Z =$ the Cosine of $(134^\circ:39':56'')$ the greater Azimuth; $x =$ the Sine and y the Cosine of the requir'd Latitude; then, since the Azimuths from the North are both obtuse, by *Trig. Theorem 8.* $Ax - ay =$ the Cosine of PA ; and $Hx - by =$ the Cosine of PB ; but since the Interval between the two Observations is but small, the Sun's Declination may be taken for the same in both, and so PA and PB esteem'd equal; wherefore $Ax - ay =$

$Hx - by$, and dividing by y , $A \frac{x}{y} - am = H$

$\frac{x}{y} - bz$, or $bz - am = H \frac{x}{y} - A \frac{x}{y}$, wherefore $\frac{x}{y}$

$= \frac{bz - am}{H - A} =$ the Tangent of the requir'd Latitude.

Whence we have gained this general Theorem, Multiply the Cosine of the Sun's Altitude at each Observation by the Cosine of the corresponding Azimuth: Divide the Difference between these two Products by the Difference of the Sines of the two Altitudes of the Sun; the Quotient is the Tangent of the Latitude of the Place of Observation; which being known, the Sun's Declination and the Hours of Observation may be found by the common Cases of Trigonometry.

By this Theorem the Latitude required is discovered to be $54^\circ:51'$, whence the Declination of

the Sun is $= 20^{\circ} 24'$, answering to the 11th Day of *May*, and the Hours of Observation 8 and 10 o'Clock in the Morning.

SCHOLIUM.

This Problem (first invented and published by Mr. *Robert Heath*) may be of Use to such as having an exact Meridian Line, with an extended Horizontal Plane to measure Azimuths upon; together with a well graduated Quadrant to take Altitudes with, but being destitute of a good Chronometer, and unwilling to confide in any Predetermination of the Latitude of the Place they are in, would observe the Ingress of the Sun into any particular Point of the Ecliptic, or the Declination of any Star, Planet or Comet, which taking the Observations as near as may be together, it performs with the Assumption of as few Principles as can be desired.



PROB.

P R O B. V.

Being Question 6. in the *Ladies Diary*, 1742. proposed by Mr. NICH. FARRER.

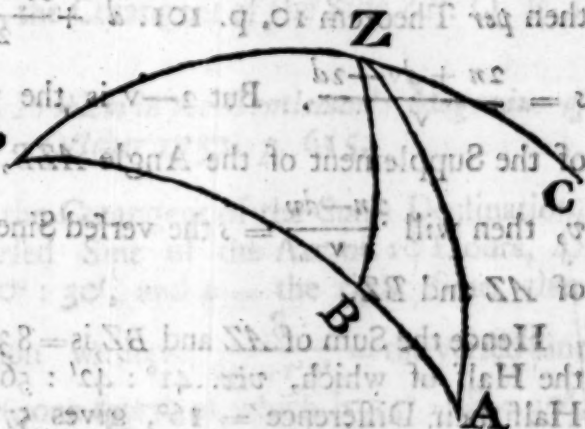
On the 20th of December, 1740. at fourteen Minutes past Eight at Night, I observed two noted fixed Stars on the Meridian, the Difference of their Meridian Altitudes being thirty-seven Degrees, and at Ten o'Clock the same Night I found their Azimuths $31^{\circ} : 40'$ and $61^{\circ} : 45'$ both West, and their Difference of Altitudes 32° ; Quere the Latitude of the Place of Observation?

SOLUTION.

SINCE

the two Stars were both observed upon the Meridian at the same time, they lie in the same Circle of De-

clination, and have the same right Ascension and consequently the Difference of their Meridian Altitudes is equal to the Difference of their Declinations or of their polar Distances; wherefore if *P* represent the elevated Pole; *Z* the Zenith; *A* the Place of one Star, and *B* that of the other at the second



second Observation ; BA will be equal to $(37^{\circ} : 00')$ the Difference of the two Stars polar Distances = the Difference of their Meridian Altitudes ; the Angle AZC the Azimuth of one Star, and Angle BZC that of the other at the second Observation, and Angle BZA ($= 30^{\circ} : 6'$) their Difference ; also since the Difference between any two Altitudes is equal to that between their corresponding Zenith Distances, the Difference of ZA and ZB will be ($= 32^{\circ} : 00'$) equal to the Difference of Altitudes at the second Observation ; wherefore in the Triangle ABZ are known $BA = 37 : 00'$, the Angle $AZB = 30^{\circ} : 5'$, and the Difference of the Sides AZ and $BZ = 32^{\circ}$, to find AZ or BZ separately, which will easiest be done by first finding their Sum, in order to which put d to represent the versed Sine of the Difference of AZ and BZ , v for the versed Sine of their Sum : Also let v represent the versed Sine of the Angle AZB , and n the versed Sine of AB ;

then per Theorem 10. p. 101. $d + \frac{sv - dv}{2} = n$

$s = \frac{2n + dv - 2d}{v}$. But $2 - v$ is the versed Sine of the Supplement of the Angle AZB , which call v , then will $\frac{2n - dv}{v} = s$ the versed Sine of the Sum of AZ and BZ .

Hence the Sum of AZ and BZ is $= 83^{\circ} : 25' : 53''$, the Half of which, viz. $41^{\circ} : 42' : 56''$ added to Half their Difference $= 16^{\circ}$, gives $57^{\circ} : 42' : 56'' = AZ$ the greater Side, and the Difference between the said half Sum and half Difference, gives $25^{\circ} : 42' : 56'' = BZ$ the lesser Side ; whence the Latitude is easily found $= 36^{\circ} : 47'$. Q. E. I.

Solution

*Solution to the Second Problem, Petersbourg Acts,
Vol. V. p. 25.*

If x and y , a and b , c and d , express the Sines and Cosines of the Sides ZP and SP , and sP respectively, and e and f , g and h , that of the Angle ZPS and ZPs ; then, by *Theorem 12* of our *Spherics*, we have $\frac{bx - ays}{ae} = \frac{dx - cyb}{cg}$, or putting p for the Sine of the Angle sPS , t the Cotangent of SP , and v that of sP , we get $\frac{tg - ve}{p} = \frac{y}{x}$ the Cotangent of the Side ZP .

Solution to the Third Problem.

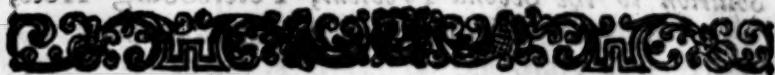
If the Letters be put to represent as in the last, then by *Theorem 8*. by $-ans = dy - cby$; therefore $\frac{y}{x} = \frac{af - cb}{b - d}$, the Cotangent of the Side ZP . Q. E. I.

*Solution to a Problem in the Gentleman's Magazine of
October 1737. p. 615.*

Put t for the Cotangent of the Sun's Declination, g = the versed Sine of the Arc of 10 Hours, 42 Min. = $160^\circ : 30'$, and n = the right, Sine, then by Calculation we have $\frac{8}{2g + t^2 n^2}$ = the versed Sine of $141^\circ : 54'$, one fourth of which is $35^\circ : 28' : 30''$, the Latitude of the Place next the Equator.

Here the Reader may observe there is little Trouble (by the Help of the foregoing Theorems) in solving the most difficult Problems that are generally proposed, a Variety of which may be seen in the *Third Volume* of this Work.

Of



Of UNLIMITED QUESTIONS, and such
as are known by the Name of
Diophantine Problems.

WE have hitherto discoursed of Problems producing Equations equal in Number to the required Quantities, and which are, for that Reason, limited, determinate, and confin'd to one Answer, except so far as the Plurality of Roots belonging to adfected Equations may hinder; and propose here to speak a little concerning such as producing Equations inferior in Number to the *Quæsitæ*, and being therefore capable of innumerable Answers, are called *Unlimited Questions*; tho' these Kind of Questions are, from the Nature of the Equations they produce, so easy to have all their Conditions satisfy'd, that they admit infinite Variety of Values of the requir'd Quantities; yet when the Solutions are requir'd in whole Numbers alone, and mix'd and broken Numbers not allow'd as pertinent Answers (as is commonly the Case) then there is something of Nicety in the Investigation of allowable Numbers. Our Method of binding down this undeterminate Species of Problems to only their integral Solutions, we shall here illustrate, rather by solving Problems taken from *Diophantus*, and others, than by formal Precepts:

PROB.

P R O B. I.

To find three Numbers whose Sum is 100 ; such that if the First be multiplied by Unity, the Second by 3, and the Third by 2, the Sum of the Three Products may make 138 ?

S O L U T I O N.

PUT x, y and z , for the three required Numbers respectively ;

Then First, $x + y + z = 100$ } by the Data.
And Secondly, $x + 3y + 2z = 138$

There being three unknown Quantities, and only two Equations resulting from the Relations mentioned in the Question, shews the Problem to be capable of innumerable Solutions ; but to discover the Values of the unknown Quantities in Integers alone, we proceed thus :

From transposing y and z in the first Equation, x is $= 100 - y - z$, and therefore by substituting its Equal instead of x in the second Equation, we have $100 - y - z + 3y + 2z = 138$; that is, when

$$\text{contracted } 2y + z = 38 ; \text{ ergo } y = \frac{38 - z}{2} = 19 - \frac{z}{2}$$

Hence that y may be an Integer $\frac{-z}{2}$, or rather $\frac{z}{2}$ must be some whole Number ; let it be n , then will $\frac{z}{2} = n$, and $z = 2n$, which put for z in the

$$\text{Equation } y = \frac{38 - z}{2}, \text{ gives } y = \frac{38 - 2n}{2} = 19 - n ;$$

consequently by adding x and the Values of y and z together, we have from the 1st Step, $x + 19 - n + 2n = 100$, whence x is $= 81 - n$.

Y

Whence

Whence $\begin{cases} x = 81 - n \\ y = 19 - n \\ z = 2n \end{cases}$ are such Values of x, y and z , as will answer the Conditions of the Question.

It may be observed from the Value of y , that n , the assumed whole Number, must be less than 19, in order to make $19 - n = y$ possible; and therefore being admitted for one Value of n ; 2 for another, &c. its Value can be varied but 18 Times; and consequently that this Question can admit of only 18 Answers in affirmative whole Numbers.

PROB. II.

To divide 96 into three whole Numbers, such that $\frac{1}{3}$ of the First, together with $\frac{1}{4}$ of the Second; and $\frac{1}{5}$ of the Third, may make 12?

SOLUTION.

PUT x, y and z for the three whole Numbers sought, then will

$$\text{And } \begin{cases} x + y + z = 96 \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 12 \end{cases} \text{ by the Question:}$$

By bringing the second Equation out of Fractions, we have $72x + 63y + 56z = 6048$. But from the 1st Equation $x = 96 - y - z$, whence $72x = 6912 - 72y - 72z$, which put for $72x$ in the 3d Equation, gives $6912 - 72y - 72z + 63y + 56z = 6048$, which becomes, after Contraction, $864 - 9y - 16z = 0$; whence $\frac{864 - 16z}{9} = y$, or which is e-

quivalent $96 - z - \frac{7z}{9} = y$; hence that y may be an Integer $\frac{7z}{9}$, or rather $\frac{7z}{9}$ must be some whole Number

Number ; suppose it k , then will $\frac{7z}{9} = k$, $7z = 9k$, and $z = \frac{9k}{7}$; whence that z may be a whole Number, it is farther requisite that k be some Multiple of 7, or which is the same, that $\frac{k}{7}$ be some integral Number ; let it be n , then will $z = \frac{9k}{7}$ be $= 9n$; wherefore by substituting $9n$ for z in $\frac{864-16z}{9} = y$, we have $\frac{864-144n}{9} = 9-16n = y$; hence writing instead of y and z , their Equals, $96-16n$ and $9n$; $x+y+z = 96$ will become $x+96-16n+9n=96$, which reduced gives $x = 7n$.

Hence $\left\{ \begin{array}{l} x=7n \\ y=96-16n \\ z=9n \end{array} \right\}$ are such Values of x , y and z , as will answer the Conditions of the Problem.

It is obvious, that to make y possible, $16n$ must be less than 96, and n than 6 ; and consequently, that this Problem can only be answered by 5 Sets of integral positive Numbers, viz.

1	n being supposed	= 1	$x = 7$	$y = 80$	$z = 9$
2	"	= 2	$x = 14$	$y = 64$	$z = 18$
3	"	= 3	$x = 21$	$y = 48$	$z = 27$
4	"	= 4	$x = 28$	$y = 32$	$z = 36$
5	"	= 5	$x = 35$	$y = 16$	$z = 45$

P R O B. III.

A owes to B a certain given Sum of Money, and has, when he comes to discharge the Debt, no other Pieces about him but Louis d'Ors (Value 17 s. each); also B is destitute of any other Coin than Guineas: What Number of Louis d'Ors must A give to B, and how many Guineas must B return to A, so as, upon the Balance, to keep in his Hands the exact Sum due to him from A?

SOLUTION.

PUT x = the required Number of Guineas, y that of Louis d'Ors, and for the Number of Shillings contained in the given Debt put S .

Then by the Conditions of the Question, we

have $17y - 21x = S$, whence $y = \frac{21x + S}{17} = x +$

$\frac{4x + S}{17}$ = by the Nature of the Thing to some integral Number ; wherefore $\frac{4x + S}{17}$ is also some In-

teger ; let it be n , then will $\frac{4x + S}{17}$ be $= n$; $4x + S$

$= 17n$, and $x = \frac{17n - S}{4} = 4n + \frac{n - S}{4}$; consequent-

ly $\frac{n - S}{4}$ or $\frac{S - n}{4}$ must be some whole Number ;

call it k ; then will $\frac{n - S}{4} = k$, and $n - S = 4k$, or

$S - n = 4k$; whence $n = 4k + S$, or $S - 4k$: Ergo

$4n + \frac{n-S}{4}$ is $= 16k + 4S + \frac{4Sk + S-}{4} = 17k + 4S$, or $4S - 17k = x$; now by substituting this

Value of x in the above Equation $\frac{21x + S}{17} = y$, we

have $\frac{357k + 85S}{17} = 21k + 5S = y$, or $5S - 21k = y$.

Whence $\begin{cases} y = 21k + 5S, \text{ or } 5S - 21k = \text{the No. of Louis d'Ors,} \\ x = 17k + 4S, \text{ or } 4S - 17k = \text{the No. of Guineas.} \end{cases}$

k being any whole Affirmative or Negative Number assumed at Pleasure, that is not inconsistent with the Expressions.

Example, Suppose the Sum wherein A is indebted to B were 5 l. or 100 s. then will $S = 100$, and taking k any Number, suppose 23, (which is the greatest Number that it can be in $4S - 17k$, or $400 - 17k = x$ for 17 times 24 would be 408, which is greater than 400, &c.) $5S - 21k$ will be $500 - 483 = 17 = y$, and $4S - 17k = 400 - 391 = 9 = x$; whence it appears that A may pay B 100 Shillings, or 5 l. by giving him 17 Louis d'Ors, B returning him back again 9 Guineas.

If k be taken equal any other Number less than 23, we shall have other Answers to the Problem, viz. the Expressions $y = 5S - 21k$ and $x = 4S - 17k$, will exhibit 23 Answers to the Question, by taking $k = 23, 22, 21, \&c.$ to 1. But the Expressions $5S + 21k = y$, or $4S + 17k = x$ will afford us an infinite Number of Answers, by taking $k = 0, 1, 2, 3, \&c. ad infinitum$.

From the Expressions for x and y , above discover'd, it may be observ'd that the Difference of these two Quantities is $4k + S$, or $S - 4k$, whence some Compendium may be found, where large Numbers are concerned, for deducing one Number from an assigned Value of the other.

PROB. IV.

A owes to B a certain Sum of Money, and has no other Coin to discharge the Debt with than Louis d'Ors, (Value 17 s. each) and Guineas; the Question is to tell how A must pay the Debt?

SOLUTION.

LET there be assumed the same as in the foregoing; then will $17y + 21x = S$, and by a like Process we get $x = 17k - 4S$, and $y = 5S - 21k$.

Example 1. Suppose the Debt 5l. or 100 Shillings, as before, then will $x = 17k - 400$, and $y = 500 - 21k$; by $17k - 400 = x$ it appears that k must be equal 24, in order that x may be Affirmative; and by $500 - 21k = y$, it appears that k cannot be greater than 23, if y be Affirmative: Hence it is evident, that it is impossible to pay 5l. in Guineas and Louis d'Ors.

Example 2. Suppose the Debt 100 l. or 2000 s. then will $x = 17k - 8000$, and $y = 10000 - 21k$; by $17k - 8000 = x$, it is evident that k must be 471, in order that x may be an Affirmative, and by $10000 - 21k = y$, it appears that k may be any Number less than 477; hence k may be 471, 472, 473, 474, 475, or 476, which shews there may be six Answers to this Problem, and no more, i. e. $477 - 471 = 6$.

Note, This last Example is the same with Case 3. of the Solution to a Problem in Dr. Saunderson's Algebra, Vol. I. p. 313.

PROB.

P R O B. V.

By Mr. ROBERT FEARNside in *Ladies Diary* 1734.

*A Lady of Wit, Youth and Beauty beside,
Remote from all Cares, but of being a Bride:
Surpriz'd her fond Lover one Morning in May,
And dispatch'd him for Parson and Licence away.
But how great her Confusion, when Strephon brought
News*

*That the Parson a Licence to grant did refuse!
Her Age, which the Lover nor Lady could tell,
Was the Cause that this fatal Disaster befel.*

*Therefore, Ladies, their bumble Request is, you'll
show
The Way how to do't by the *Data below:
And this by a general Method explain,
So that Lovers may never be non-plus'd again.*

S O L U T I O N.

PUT x for the Year of Christ when the Lady was born, and $S=18$, $m=8$, $r=10$; then per the Nature of the Question, $\frac{x+9-S}{28} = a$, $\frac{x+1-m}{19} = b$ and $\frac{x+3-r}{15} = c$ will be whole Numbers, and by Reduction we get $x=4845S-3780m-1064r-4713-7980k$.

Whence this Canon or Rule.

Multiply the Golden Number by 3789, the Roman Indiction by 1064, and the Sum of the

* Cycle of the Sun 18. Golden Number 8. Roman Indiction 10, that Year the Lady was born.

two Products, and 4713; taken from 4845 Times the Cycle of the Sun, will leave a Remainder, which divided by 7980, the Quotient will be the Value of k , and the Remainder, the Year of Christ when the Lady was born. Q. E. I.

If any Person please to reduce $\frac{x+9-18}{28} = a$, $\frac{x+1-8}{19} = b$, and $\frac{x+3-1}{15} = c$, they will find the Value of $x = 1717 + 7980k$; therefore if $k=0$ $x=1717$ as above.

Also it may be further observed, that the Reduction of $\frac{x+9-5}{28} = a$, and $\frac{x+1-m}{19} = b$ will give the following general Rule for finding the *Dionysian* Period, when we have the Cycle of the Sun and Moon given.

The R U L E.

Multiply the Cycle of the Sun by 57, and the Cycle of the Moon by 56; divide the Difference of their Products by 532, the Remainder, after the Quotient, will give the Year of the *Dionysian* Period.

Note, *The Solution of this Problem, and others of a like Nature, shall be fully explain'd in the third Volume.*

PROB. VI.

Suppose a Vintner would make a Mixture consisting of 40 Gallons, viz. of Canary at 7 s. Sherry at 5 s. and White Port at 4 s. a Gallon; how many Gallons of each Sort must be put together, that he may sell the Mixture at 6 s. a Gallon without Gain or Loss by such Sale?

SOLUTION.

THE Vintner by the Rule of Alligation, and the Help of the Rule of Proportion, finds he must put together 24 Gallons of Canary, 8 Gallons of Sherry, and 8 Gallons of White Port. But now, if this Solution does not answer the Vintner's Intent; that is, suppose he would have his Mixture otherwise proportion'd, I then say he must have Recourse to Algebra for Assistance: For 'tis by an algebraic Process only, that all the Solutions possible can be discover'd; which wonderful Advantage I shall shew in an algebraic Solution of this Prob.

Let $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$ be = the Gallons of $\begin{Bmatrix} \text{Canary,} \\ \text{Sherry,} \\ \text{White Port.} \end{Bmatrix}$

Then $x + y + z = 40$
And $7x + 5y + 4z = 240$ } by the Hypothesis

Thence $z = 40 - x - y$, and $z = \frac{240 - 7x - 5y}{4}$,

therefore $160 - 4x - 4y = 240 - 7x - 5y$ and $3x =$

$80 - 3x$; $\therefore x = \frac{80 - y}{3}$, hence $\frac{2 - y}{3}$ or $\frac{y - 2}{3}$

must be some whole Number, which call n , then will
Z y - 2

$\frac{y-2}{3} = n$ and $\therefore y = 3n + 2$, which put for y above makes $x = \frac{80-3n-2}{3} = 26 - n$, and by writing the Values of y and x in the Equation $x + y + z = 40$ makes $26 - n, + 3n + 2, + z = 40$, that is, $28 + 2n + z = 40 \therefore z = 12 - 2n$.

Hence the Gallons of $\left\{ \begin{array}{l} \text{Canary} = 26 - n \\ \text{Sherry} = 3n + 2 \\ \text{White Port} = 12 - 2n \end{array} \right\}$

n being any affirmative whole Number that is not inconsistent with the Expressions above; *i. e.* n may be 1, 3, 3, 4 or 5 (but not 6, for if $n=6$ then $12-2n$ would be $12-12=0$) whence the Number of Gallons of

Canary	Sherry	White Port
25	5	10
24	8	8
23	11	6
22	14	4
21	17	2
20	20	0

Thus we have obtain'd a perfect Solution of this Proposition, having determin'd all the possible Means of forming the Mixture.

PROB.

P R O B. VII.

Being the 8th of the second Book of Diophantus's Arithmetick.

It is required to divide a given Square Number into two such Parts, that each may be a Square Number.

N. B. By Numbers in a simple Sense, are meant by Diophantus all rational Numbers, whether they be whole Numbers or Fractions, in contradiction to Surd Roots, and all such other Numbers as are incommensurable to Unity, which are commonly called irrational Numbers.

SOLUTION.

LET bb be the Square Number proposed to be divided, and since the two Parts of this Number are also each to be Squares, let xx be one of them, and the other will be $bb - xx$, which, as well as the former, is to be a Square. Now as the Problem supposes no Relation betwixt the Sides of these two Squares, we are left at Liberty to feign what Relation we please betwixt them, provided that will but furnish us with proper Equation for determining the Sides of the required Squares: Let us then suppose the Side of the second Square to be $b - dx$, then must the Square of $b - dx$ be equal to the second Square, that is, to $bb - xx$; but the Square of $b - dx$ is $b^2 - 2bdx + d^2x^2$, therefore $b^2 - 2bdx + d^2x^2 = bb - xx$; resolve this Equation by striking off b^2 from both Sides, and expunging the common x , and you will have $x = \frac{2db}{1 + d^2}$ the Root

of the first Square, and $b - dx = b - \frac{2bd^2}{1+d^2} =$
 $\frac{b-bd^2}{1+d^2} = b \times \frac{1-d^2}{1+d^2} =$ the Side of the other Square.

Hence if the arbitrary Quantity d be equal 1, then will b and o be the Sides of the two Squares, and bb and o the required square Numbers themselves.

If d be assumed = 2, then will $\frac{4b}{5}$, and $-\frac{3b}{5}$ be the Sides of the two Squares, and $\frac{16bb}{25}$ and $\frac{9bb}{25}$ the Numbers sought.

If $d = 3$, then $\frac{6b}{10}$ and $-\frac{8b}{10}$ are the Sides of the two Squares, and the Numbers required $\frac{36bb}{100}$ and $\frac{64bb}{100}$.

If $d = 4$, then $\frac{8b}{17}$ and $-\frac{15b}{17}$ are the Sides of the two Squares, and $\frac{64bb}{289}$ and $\frac{225bb}{289}$ the Numbers themselves.

Whence if the Square Number given to be divided were 100, the two Square Numbers required would be found 64 and 36, when d is assumed either 2 or 3, $\frac{6400}{289}$ and $\frac{22500}{289}$, when d is taken = 4, &c. by taking different Values of d , we get as many Answers as we please.

SCHOLIUM.

Since $\frac{2db}{1+d^2}$ and $\frac{b-bd^2}{1+d^2}$ are discovered to be the Sides of the required Squares; that is, since the Sum of the Squares of $\frac{2db}{1+d^2}$ and $\frac{b-bd^2}{1+d^2}$ is found to be $= bb$; and since (from *Euclid* 1. 47.) when the Sum of the Squares of any two Numbers is equal to the Square of some other Number, those three Numbers form a Right-angled Triangle; it follows that if we make as many Right-angled Triangles as we please, having the common Hypotenuse b , the Legs of any one of these Triangles will be the Roots of the Squares we define, and their Squares the required Squares themselves.

Also, To find as many Right-angled Triangles as we please, having all of them one given Hypotenuse; which is commonly made a distinct Problem, has its Solution involved in this Reasoning, viz.

Let the given common Hypotenuse be represented by b , and let the Value of d be assumed at Pleasure; then $\frac{2db}{1+d^2}$ is one, and $\frac{b-bd^2}{1+d^2}$ the other Leg of a Right-angled Triangle having that Hypotenuse. As in the following Table.

Value

Value of d .	Legs of the Triangle.	Hypothenuse.	Values of d .	Legs.	Hypothenuse.
2	$\frac{4b}{5}$ and $\frac{3b}{5}$	b	9	$\frac{9b}{41}$ and $\frac{40b}{41}$	b
4	$\frac{8b}{17}$ and $\frac{15b}{17}$	b	&c.	&c. and &c.	&c.
5	$\frac{5b}{13}$ and $\frac{12b}{13}$	b	$\frac{3}{7}$	$\frac{21b}{29}$ and $\frac{20b}{29}$	b
6	$\frac{12b}{37}$ and $\frac{35b}{37}$	b	$\frac{3}{8}$	$\frac{48b}{73}$ and $\frac{55b}{73}$	b
7	$\frac{7b}{25}$ and $\frac{24b}{25}$	b	$\frac{3}{10}$	$\frac{60b}{101}$ and $\frac{99b}{101}$	b
8	$\frac{16b}{63}$ and $\frac{61b}{63}$	b	$\frac{3}{11}$	$\frac{33b}{65}$ and $\frac{56b}{65}$	b
			&c.	&c. and &c.	

It is evident, from observing the Values of the Sides of the Triangle, that when b is such a Number as can be divided by the Denominators, the Legs will be integral; whence several Triangles may be found in whole Numbers having a common Hypothenuse; as if the Hypothenuse were 65, then 52 and 39, 25 and 60, 16 and 63, 33 and 56, would be the Legs of four Triangles having that Hypothenuse common.

One Thing farther may be worth observing in this *Scholium*, viz. that since $\frac{2bd}{1+d^2}$ and $\frac{b-bd^2}{1+d^2}$ represent the Legs of that rectangular Triangle whose Hypothenuse is b , i. e. since $\frac{2db}{1+d^2}$, $\frac{b-bd^2}{1+d^2}$ and b are the three Sides of a Right-angled Triangle; and since the Sides of any Triangle being all augmented or diminished in one and the same Proportion give the Sides of another Triangle similar to the first; if we say, as b : to $1+d^2$:: so is each Side of the general rectangular Triangle just mentioned : to fourth; we shall have second, $1-d^2$ and $1+d^2$ for

for the three Sides of another Right-angled Triangle, similar to the former. But because to make $1-d^2$, one of the Legs, possible, it is requisite that d should be some fractional Number, let it be =

$\frac{m}{p}$; then will the three Sides be $1 + \frac{mm}{pp}$, $\frac{2m}{p}$ and $1 -$

$\frac{mm}{pp}$; which being reduced to a common Denomi-

nator become $\frac{pp+mm}{pp}$, $\frac{2mp}{pp}$ and $\frac{pp-mm}{pp}$; and mul-

tiplying each of these by pp (which being common will not prejudice the Similarity) we have $pp+mm$, $2mp$ and $pp-mm$, for the three Sides of the Triangle in whole Numbers. Hence we draw this

THEOREM.

Take any two Numbers at Pleasure, their double Rectangle shall be one Leg, the Difference of their Squares the other, and the Sum of their Squares the Hypothenuse of a Right-angled plain Triangle.

This Theorem may easily be demonstrated in the synthetical Manner, by adding together the Squares of $pp-mm$ and $2mp$, and comparing their Sum with the Square of $pp+mm$.



PROB.

P R O B. VIII.

Being the 10th of the second Book of Diophantus.

To divide the Sum of two Squares (as $aa + bb$) into two other Squares.

SOLUTION.

LET $x + a$ = the Side of one of the Squares sought, and $b - dx$, that of the other; then will $xx + 2xa + aa$ be one of the required Squares, and $bb - 2bdx + ddxx$, the other; wherefore $xx + 2xa + aa + bb - 2bdx + ddxx$ is $= aa + bb$ = the Sum of the two given Squares; whence, by expunging Quantities common to both Sides of the Equation, we have $xx + 2xa - 2bdx + ddxx = 0$, and

$$xx + 2xa + ddxx = 2bdx; \text{ and dividing by } x, \\ x + 2a + dd x = 2bd; \text{ wherefore } x = \frac{2bd - 2a}{1 + dd}$$

$$\text{and } x + a = \frac{2bd - 2a}{1 + dd} + a, = \frac{2bd - 2a + a + add}{1 + dd}, \\ = \frac{2bd - a + ad^2}{1 + dd} = \text{the Side of one of the Squares,}$$

$$\text{and } b - dx = b - \frac{2bdd + 2da}{1 + dd} = \frac{b + bdd - 2bdd + 2da}{1 + dd} \\ = \frac{b - bdd + 2da}{1 + dd} = \text{the Side of the other.}$$

From these two Equations, the Values of a and b being given, as many different Answers to this Problem may be obtain'd as we please to assume different Values of the arbitrary Quantity d .

As

As if d be assumed equal 1, then b and a , the Sides of the given Squares, will also be the Sides of those required: If d be = 2, then $\frac{3a+4b}{5}$ and $\frac{4a-3b}{5}$ will be the Sides of the two Squares. If d be = 3, then $\frac{8a+6b}{10}$ and $\frac{6a-8b}{10}$; if = 4, then $\frac{15a+8b}{17}$ and $\frac{8a-15b}{17}$; if = 5, then $\frac{24a+10b}{26}$ and $\frac{10a-24b}{26}$ are the Sides of the desired Squares. And for other Values of d .

EXAMPLE.

Let $aa+bb$ be given = 13, or $aa=9$, and $bb=4$; i. e. $a=3$ and $b=2$; to divide this Sum into two other Squares.

Then if d be taken = 2, $\frac{17}{5}$ and $\frac{6}{5}$ are the Sides of the required Squares, and $\frac{289}{25}$ and $\frac{36}{25}$ the Numbers sought; which will answer the Condition

of the Problem, for $\frac{289}{25} + \frac{36}{25}$ is = $\frac{325}{25} = 13$. A-

gain, if d be = 3, then $\frac{36}{10}$ and $\frac{2}{10}$, or $\frac{18}{5}$ and $\frac{1}{5}$

are the Sides of the desired Squares, and $\frac{324}{25}$ and $\frac{1}{25}$

the required Squares themselves; which are also rightly determined, for $\frac{324}{25} + \frac{1}{25}$ is = $\frac{325}{25} = 13$.

Once more, if d be supposed = 4, then $\frac{61}{17}$ and $\frac{-6}{17}$

are the Sides of the Squares sought, and $\frac{3721}{289}$ and

$\frac{36}{289}$ the Square Numbers themselves; which are also truly investigated, for $\frac{3721}{289} + \frac{36}{289}$ is $= \frac{3757}{289} = 13$, and thus we may proceed *ad infinitum*, by varying the Value of d .

SCHOLIUM.

Since we have found the Side of one of the required Squares $= \frac{2bd - a + add}{1 + dd}$ and of the other, $\frac{b - bdd + 2da}{1 + dd}$; if we make the Side of one of the given Squares (suppose a) $= 0$, so that it may vanish, and thereby reduce this Problem to that of having a square Number given to be divided into two other Squares; by blotting out all the Terms affected with a , we have $\frac{2bd}{1 + dd}$ for the Side of one of the Squares, and $\frac{b - bdd}{1 + dd}$ for that of the other; which are the same Values we discovered in *Prob. 7*. Wherefore it from hence appears, that the Solution of *Prob. 7* might have been drawn from that of the present Problem, by Way of Corollary; but only that we imagined an independant Solution thereto might be more agreeable to such of our Readers as had not been much conversant in these Matters.

PROB.

P R O B. IX.

To find two Numbers such, that the Square of the First being added to the Second, may be equal to the Square of the Second when added to the First.

S O L U T I O N.

PU T x = to one of the required Numbers, and taking n of any Value at Pleasure, let nx be the other; then we shall have, by the Nature of the Problem, $xx + nx = nxx + x$, which divided by the common x , gives $x + n = nnx + 1$, and so

$x = \frac{n-1}{nn-1}$; but $\frac{n-1}{nn-1}$ is found by Division =

$\frac{1}{n+1}$; Ergo $x = \frac{1}{n+1}$ one of the Numbers, and

$nx = \frac{n}{n+1}$ = the other; wherefore the Sum of

these Numbers is $\frac{1}{n+1} + \frac{n}{n+1} = \frac{n+1}{n+1} = 1$.

Wherefore,

If Unity be broken into any two Parts at Pleasure, those two Parts shall be Numbers satisfying the Conditions of this Problem.

E X A M P L E.

1. Let Unity be broke into two Parts $\frac{1}{4}$ and $\frac{3}{4}$

then $\frac{1}{16} + \frac{3}{4} = \frac{13}{16}$ and $\frac{9}{16} + \frac{1}{4}$ is also = $\frac{13}{16}$; where-

fore the Fractions $\frac{1}{4}$ and $\frac{3}{4}$ will answer the Question.

A a 2

2. Let

2. Let Unity be broke into two Parts $\frac{2}{5}$ and $\frac{3}{5}$;
 then $\frac{4}{25} + \frac{3}{5}$ is $= \frac{19}{25}$, and so is $\frac{9}{25} + \frac{2}{5} = \frac{19}{25}$;
 wherefore the Numbers $\frac{2}{5}$ and $\frac{3}{5}$ may be admitted
 as Answers; and thus we may proceed at Plea-
 sure.

PROB. X.

By Mr. JOHN MAY, Jun. of Amsterdam.

There came three Dutchmen of my Acquaintance to see me, being lately married; they brought their Wives with them. The Men's Names were Hendrick, Claas, and Cornelius; the Women's Geertruij, Catriin, and Anna: But I forgot the Name of each Man's Wife. They told me they had been at Market to buy Hogs; each Person bought as many Hogs as they gave Shillings for one Hog; Hendrick bought 23 Hogs more than Catriin; and Claas bought 11 more than Geertruij; likewise, each Man laid out 3 Guineas more than his Wife. I desire to know the Names of each Man's Wife?

SOLUTION by Mr. J. HILL of Derby.

CALL the Number of Hogs any Woman bought x ; the Number her Husband bought $x+n$; Money laid out by the Woman is x^2 Shillings; Money laid out by the Husband is $x^2 + 2nx + n^2$ Shillings. Equation $x^2 + 2nx + n^2 = x^2 + 63$.

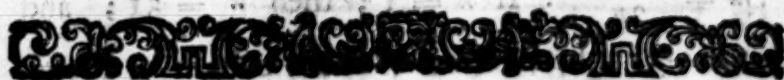
$\therefore x = \frac{63-n^2}{2n}$: If $n = 1$; then $x = 31$; and
 $x + n = 32$; hence some Woman bought 31 Hogs,
 and her Husband 32 ; if $n = 3$ then $x = 9$ and
 $x + n = 12$; therefore some other Woman bought
 9, and her Husband 12 : If $n = 7$ then $x = 1$ and
 $n + x = 8$. *Ergo* some Woman bought 1, and her
 Husband 8. Consequently

Hendrick bought 32, and his Wife	Anna	31	
Claas	12	Catriin	9
Cornelius	8	Geertruij	1

Note, A more Variety of these kind of Problems, may be seen in Vol. 2. of Dr. *Saunderson's* excellent Treatise of *Algebra*.



QUES



QUESTIONS in the *Gentleman's* *Diary* 1743.

QUESTION. I.

By Mr. ROBERT HALL of *Boston*.

One Morning in May, I went to survey
As soon as bright Sol I espy'd;
I measur'd round a four-corner'd Ground; *Ch. Lin.*
P'tb' Margin's the Length of each Side : $AB = 1560$
The Angle at B, together with D, $BC = 1320$
An hundred and eighty Degrees, $CD = 1000$
The Meadow's Content is all that I want; $DA = 2600$
Assist me, kind Sirs, if you please.

SOLUTION by Mr. T. PEAT of Nottingham.

$AB + BC$ squared Minus $AD - DC$ squared;
divided by $2AB \times BC + 2AD \times DC$ is $= \frac{56}{91}$
(when abbreviated to the lowest Terms) and $2 -$
 $\frac{56}{91} = \frac{126}{91}$ drawn into $\frac{56}{91}$ makes $\frac{7056}{91^2}$ whose square
Root is $\frac{84}{91}$, whose half $\frac{42}{91}$ drawn into $AB \times BC +$
 $AD \times DC$ is $215,04 = 21,04$ Acres, which is the
Content of the Field $ABCD$. Q. E. I.

QUES-

QUESTION. II.

By Mr. J. RICKERBY of Saffron Waldon.

Being at Sea on a clear Day and calm, having the Prospect of a Church and a Windmill, I made an Observation, and found the Church to bear due East; and the Mill to bear NNW; and 9 Miles, or 3 Leagues, as near as I could determine. The Church's Distance from each in one Sum was 18 Miles or 6 Leagues. Quære each Distance separately, with the Bearing of the Mill from the Church?

SOLUTION by Mr. R. BEIGHTON of Brampton.

PUT $s = 18$ Miles, $b = 9 =$ the Distance betwixt the Mill and Place of Observation, $q =$ the Cosine of the Angle which is formed by Lines supposed to be drawn from the Mill, and from the Church to the Place of the Observer, Radius $= 1$. Then as $s + bq : s + b :: s - b : 11,331 =$ twice the Distance betwixt the Church and Place of Observation. Hence the Mill bears from the Church N. E. $\frac{1}{4}$ E. nearly, and the Church from the Mill S. W. $\frac{1}{4}$ W. *ferè*.



QUES.

QUESTION III.

By GAMSTON RETFORDIENS.

In two Northern Latitudes, both under the same Meridian, the one being as far from the Equator as the other from the Pole, on June the 10th the Sun was observed to rise at that Place in the greater Latitude, 1 Hour and 18 Minutes sooner than at that in the lesser. Required the Latitude of both Places.

SOLUTION by Mr. WATTS of Comb.

ON June the 10th the Sun's Declination is $23^{\circ} : 29'$, the Tangent of which call t , 1 Hour and 18 Minutes = $19^{\circ} : 30'$, whose versed Sine call v ; then will $2t^2 + v = 4348808$ which answers in the Tables of versed Sines to $55^{\circ} : 35' 21'' : 57'''$ whose half $27^{\circ} : 47' : 40'' : 58'''$ added to the half of $19^{\circ} : 30' (= 9^{\circ} : 45')$ gives $37^{\circ} : 32' 40'' : 58'''$ = the ascensional Difference in the greater Latitude. And as Tangent $23^{\circ} : 29'$ is to Sine $37^{\circ} : 32' : 40'' : 58'''$, so is Radius to Tangent $54^{\circ} : 30' : 42'' : 36'''$ the greater Latitude. Q. E. I.

QUES-

QUESTION IV.

By Mr. WILLIAM CHAPPLE.

Suppose a Hole perforated thro' the Earth in a right Line perpendicular to its Surface in Lat. 51° and that a Stone be let fall into it, so as to go on in its Descent without Interruption; where would it stop? and how many English Miles would it run thro'? Suppose it not to go beyond the Place of its final Rest*; and supposing the Earth to be an Oblate Spheroid, whose Diameter at the Equator is 7974, and the Length of its Axis 7940 English Miles.

SOLUTION by the Proposer.

PUT $2t = 7974$, $2c = 7940$, $m =$ the Sine, and n the Cosine of 51° the given Latitude;

Radius $= 1$, then will $\sqrt{c^2m^2 + c^2n^2} + \frac{c^2n^2 - c^2m^2}{\sqrt{c^2m^2 + c^2n^2}}$

$= 3976,88$ Miles, the Space run thro' by the

Stone; where it rests at the Distance $\frac{nm^2 - nmc^2}{\sqrt{c^2m^2 + c^2n^2}}$

$= 16,4$ Miles from the Center of the Earth.

Q. E. I.

* This I mention, because some think a Stone thus descending, by the Force of its Gravity, would go beyond the Center, and returning again, continue for some Time moving backward and forward, like the Swinging of a Pendulum, till it rest in its proper Place.

B b

QUES.

QUESTION V.

By Mr. RICHARD HANCOCK.

Given the Equation of a Curve AEm, $ax=y^4$;
and the Equation of another Curve AFm
 $bx^3=y$; and also Am=15, and the Area of
the Curve Space AEmF, =59,4 Quære the
Abscissa Ap, and the Ordinate pm.

SOLUTION by Mr. WILLIAM DANIEL of
Langley.

PUT $n=15 = Am$, $n=59,4 =$ Area of the
Space AEmF, then will $x = \sqrt[4]{\frac{n^2}{4} - \frac{10m}{11}} +$
 $\sqrt[4]{\frac{n^2}{4} + \frac{10m}{11}} = 12$, and $y = \sqrt[4]{\frac{n^2}{4} - \frac{10m}{11}}$
 $-\sqrt[4]{\frac{n^2}{4} + \frac{10m}{11}} = 9$. Q. E. I.

Note, the Area of the Curve Space, whose Equa-
tion is $bx^3=y$, is $\frac{y^{\frac{4}{3}}}{4}$. And that whose Equation
is $ax=y^4$, $\frac{4yx}{5}$. Or universally $\frac{zyx}{s+t} =$ Area of that
Curve, whose Equation is $ax^s=y^t$.

QUES-

QUESTION VI.

By Mr. J. THOMPSON of Witherley.

It was my Chance to be surveying a Piece of Land in the Form of a Rectangled Triangle, whose Sides I measured, and found the Hypotenuse just 30 Chains; but Night approaching, found I had blunderingly taken the other two Sides, both in one Sum, just 42 Chains 36 Links; yet I hope some ingenious Gentleman will from these Data find me the Sides and Angles, by a simple Equation?

SOLUTION by Mr. WILLIAM WHITEHEAD.

SAY as 30 Chains : 42,36 Chains :: Cosine 45° : Cosine $3^{\circ} : 12' : 22''$ = half the Difference of the acute Angles; therefore $45^{\circ} + 3^{\circ} : 12' : 22'' = 48^{\circ} : 12' : 22''$ is the greater of the acute Angles, and 45° minus $3^{\circ} : 12' : 22'' = 41^{\circ} : 47' : 58''$ the Less: Again, say as Radius : 30 Chains :: Sine of $41^{\circ} : 47' : 38''$: 19,99359 Chains the lesser Leg of the Triangle $\therefore$ the greater is 22,36641 Chains.
Q. E. I.

Note, For a Demonstration of this Solution, see p. 13.

QUESTION VII.

By Mr. PAUL SHARP.

Suppose the Distance betwixt London and York be = 150 Miles; from which two Places, two Travellers A and B set out at the same Time to meet one another; A sets out from London and travels 5 Miles the first Day, 10 the 2d, 20 the 3d, &c. in a Geometrical Progression, whose common Ratio is 2; B sets out at the same Time from York, and goes 6 Miles the first Day, 10 the 2d, 14 the 3d, &c. in Arithmetical Progression, whose common Excess is 4: Required the Time of their Meeting, and how many Miles each Man will then have travelled?

SOLUTION by Mr. W. DANIEL of Langley.

LET x = the Number of Days sought, then
 $5 \times 2^{x-1}$ = the last Term, by the Nature of Geometrical Progression, and $\frac{5 \times 2^x - 5}{2 - 1}$ = the Miles

A goes in all (See *Ward* p. 189) and by the Laws of Arithmetical Progressions $2x^2 + 4x$ the Miles
 B goes in all (See *Ward*, p. 186) $\therefore 5 \times 2^x - 5 + 2x^2 + 4x = 150$ per Question, which by Transposition and Division, gives $2^x + 0,4x^2 + 0,8x = 31$, solv'd $x = 4,3258$ the Number of Days sought. Hence A travell'd 95,27107 Miles, and B 54,72893.

QUES.

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QUESTION—VIII.

By Mr. WILLIAM BUTCHER.

A Gentleman knowing that his Servant brewed him 51,54863 Bushels of Malt at a Time (allowing 2150,42 Inches to one Bushel) is very desirous to have an Hopper to his Malt-Mill made in the Form of a Square Pyramid, but his Workman being very scarce of Boards (and no Mathematician) desires some Gentleman will be so kind as to shew him how to make it, so as to take the least Breadth of Boards, for the perpendicular Length of the slant Side the least possible.*

SOLUTION by T. SANDALL of Solyhull.

PUT $s = 110851,2$ = the Inches in 51,54863 Bushels, and say $8 : 3 :: s : \frac{3s}{8} = 41569,2$ the Cube of the Perpendicular Height of the Pyramid, whose Logarithm is 4,6187714 ∴ 1,5395904 is the Logarithm of 43,641 the Altitude sought: Again, $1 : \sqrt{3} :: 34,641 : 60$ the Perpendicular Length of the Slant-Side sought.

DEMONSTRATION.

Put x = the perpendicular Height of the Pyramid, and z = the perpendicular Length of its slant

* To hold the same Quantity.

Side,

Side, then will $n^2 - x^2 =$ the Square of the Semi-Base; which drawn into $4x$, gives $4n^2x - 4x^3 = 3s$

$\therefore 4n^2 = \frac{3s + 4x^3}{x}$, which *per* Question must be a

Minimum, whose Fluxion $\frac{8xx^3 - 3s^2x}{x^2} = 0 \therefore 8x^3 -$

$3s = 0$, and $8x^3 = 3s$, or as $8 : 3 :: s : x^3$. Again,

by writing $\frac{2s}{2}$ for $4x^2$ in the Equation above we get

$4n^2x - \frac{3s}{2} = 3s \therefore 8n^2x = 9s$; but $8x^3 = 3s \therefore$

$8n^2x = 9s = 24x^3 \therefore n^2 = 3x^2 \therefore n = x\sqrt{3}$ or as $1 : \sqrt{3} :: x : n$. Q. E. I.

QUESTION IX.

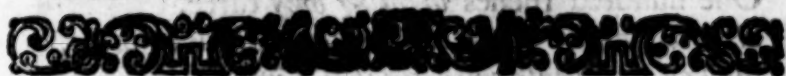
By Mr. JAMES HEMINGWAY.

In the Parallelopleuron ABCD, there is given
 (b) = the greater Base (c) = the lesser Base,
 (d) = AC = DB; and the Ratio of the
 Trapezium to the Triangle, as (r) to (s);
 thence to find the Distance of the Center of
 Gravity of the Triangle, from the Center of
 the Base AB.

SOLUTION by the Proposer:

THE Investigation of a General THEOREM would be too long for this Place, but if $AB = 12,5$. $AC = 16,75 = BD$, and $CD = 9,25$: then will $4,081942$ be the Distance sought.

QUES.



QUESTIONS in the *Ladies* Diary, 1743.

QUESTION. I.

By Mr. THOMAS COWPER.

*When that immense amazing Orb of Day
Gilded the Northern Tropic with his Ray,
From eastern Skies emitting lucid Streams,
And spread his Radiance in prolifick Beams ;
In pleasing Solitude I did repair
To view the Fields, and breathe in purer Air ;
Where joyous Birds stretch'd forth their tuneful Throats,
And pierc'd the yielding Air with thrilling Notes :
On verdant Sprays the Thrush and Blackbird sings,
While warbling Larks display'd ærial Wings ;
Artful ascending through the elastick Way,
And hail'd triumphant the solstitial Day.*

*Each Object round a grateful Scene did yield,
While teeming Plenty crown'd the blooming Field,
And beauteous Verdure deck'd the enamell'd Plain,
And pearly Dews bung on the rip'ning Grain ;
The fragrant Flowers with diff'rent Colours dy'd,
On smiling Ground display'd their gaudy Pride :
The Air with light Effluvia did abound,
Which spread their aromatic Odours round :
The Winds were hush'd, and Zephyr's gentle Breeze
Scarce heard to murmur through the shady Trees.*

*To one of them a Bough did appertain,
Whose Shadow I observ'd upon the Plain ;*

And

*And found its Distance from the Bough to be
One hundred Inches wanting only three.*

*As towards the South bright Phoebus journey'd on,
And on the Earth with greater Lustre shone;*

While up the Skies progressive Time he led,

Seventy-six Minutes and one Fifth were fled.

It happen'd that another Branch I found,

Whose Altitude above the level Ground

Was Thirteen Inches and just eight Tenths more

Than that which I observ'd the Time before;

Yet (as the first) its Distance from the Shade

No more than Ninety-seven Inches made;

And th' Base was shorter at this second View,

By one Foot and Six Tenths of One Inch too.

*Now from the Data 'tis required to shew
Their Height *, the Time and Latitude also?*

SOLUTION by Mr. J. BETTS of Oxford.

THEOREM, as $13,8 : 12,6 ::$ Radius :
Tangent of $42^\circ : 23' : 50'' = A$; and as
 $194 (= \text{twice } 97) : 13,8 ::$ Secant of $42^\circ : 23' : 50''$
: the Sine of $5^\circ : 31' : 40'' = B$. Then $A - B =$
 $36^\circ : 52' : 10'' =$ the Sun's Altitude at the first
Observation, and $A + B = 47^\circ : 55' : 30'' =$ the
Altitude at the Second. Whence, per Spherics, the
Latitude $52^\circ : 20'$ the Time of Observation
 8^h and $9^h : 16'$. Q. E. I.

The Perpendicular Height of the End of each Bough,
from which the Shadows were projected and measured.

N. B. That 17 Minutes must be deducted from the Sun's Altitude at the Time of the first Observation, and $16\frac{1}{2}$ at the Second, as an Allowance for his Semidiameter and Refraction.

Q U E S

QUESTION II.

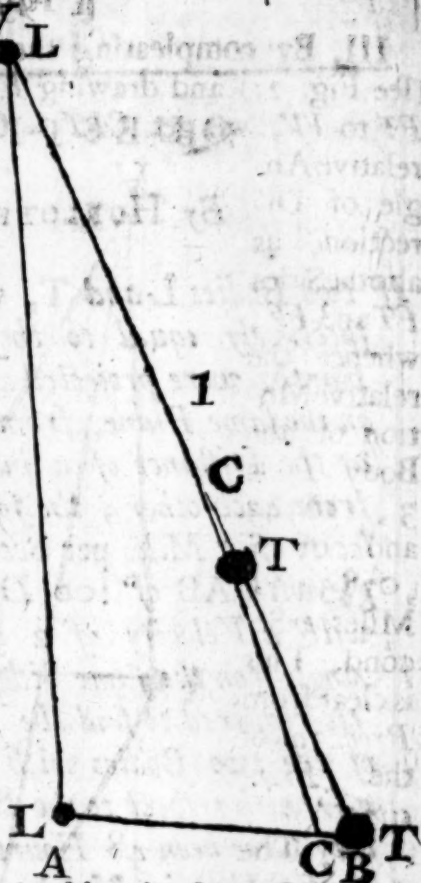
By HURLOTHRUMBO.

If two Bodies L and T, whose Masses are respectively equal to those of the Moon and Earth, were projected at the same Time, and in the same Plane, from two Places A and B, at the Distance of a hundred thousand Miles from each other; the former, L, with a Velocity of 5 Miles per Second, making an Angle with AB of 100 Degrees, and the latter with a Velocity of 2 Miles per Second, an Angle (on the same Side AB) of 60 Degrees; 'tis required to find the Distance and Position of the two Bodies with respect to each other, also with respect to the Points A and B, after they have been 48 Hours in Motion, supposing them, when in Motion, to be only acted upon by each other.

SOLUTION by Mr. J. BETTS of Oxford.

IT would take up too much Room to insert the Process at large; I shall therefore only explain the Method of Calculation:

I. In the annexed Scheme, let L and T represent the Bodies as projected, whose Masses are respectively equal to the Moon and Earth or in the Ratio of Unity to 39,778, and let AB be divided in that Ratio; then will the Point C represent the Center of Gravity (by Coroll. 4. of the *Principia* to the Laws of Motion) and its Distance from each Body will be known; from L 97547,7, and from T 2452,3 Miles.



II. Suppose the Bodies now projected (and by the same Corollary) we get the uniform Motion of the Center of Gravity, as also of the Space that moves uniformly with it, and with the same Velocity 2,3342 Miles per Second, as also the Path of the Center CC , and the Angle CCT , which gives its Position = $118^{\circ}, 58'$.

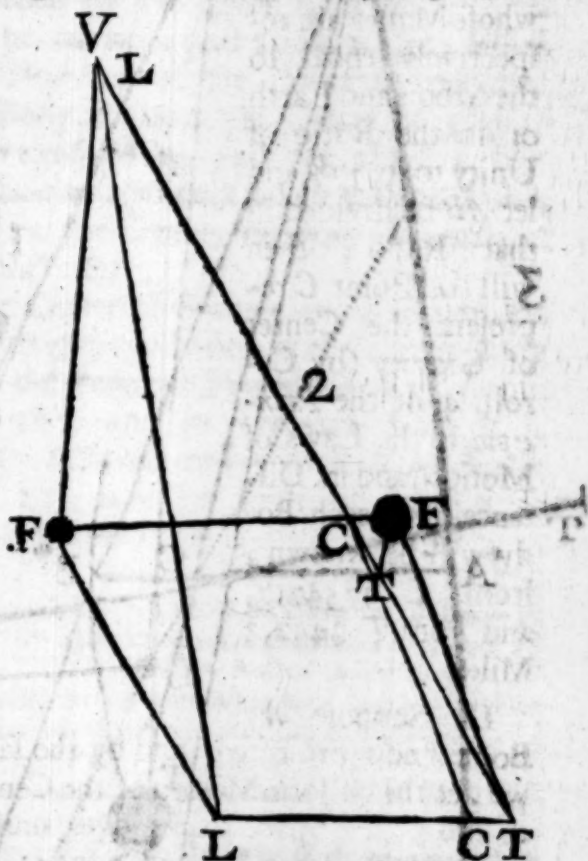
III. By

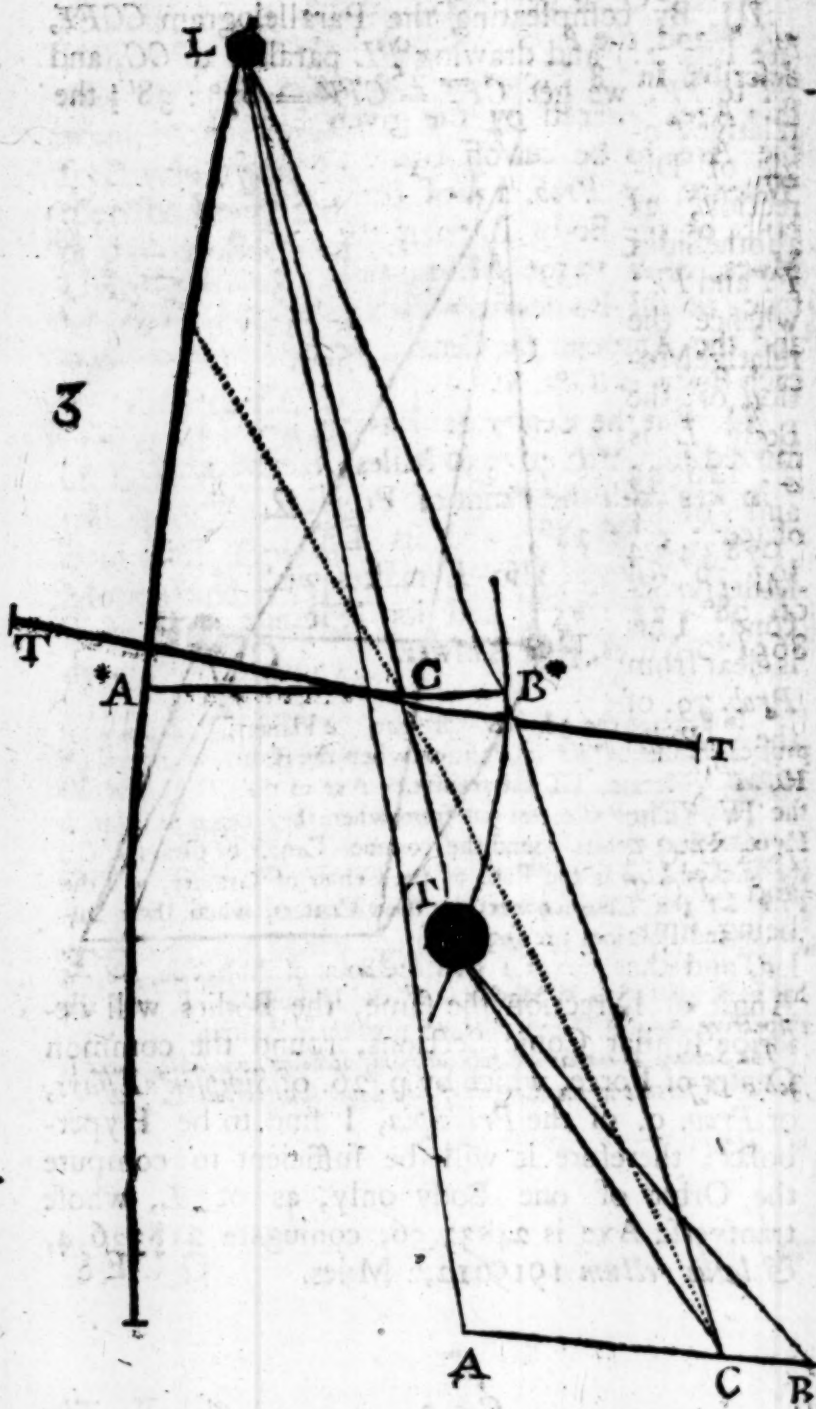
III. By completing the Parallelogram $CCFV$, (see Fig. 2.) and drawing FL parallel to CC , and FT to FV , we get $CFT = CFV = 87^\circ : 38' \frac{1}{2}$ the

relative Angle of Direction, as also the Sides FT and FV ; whence the relative Motion of the Body L is 3, 116343 and of $\mathcal{P}$, 0783434 Miles per Second. This is clear from *Prob.* 39. of the *Principia*.

IV. The Δ 's FCL and FCT being similar, and the

Angle of Direction the same, the Bodies will describe similar Conic Sections, round the common Center of Force, which by p. 26. of *Simpson's Essays*, or *Prob. 9. of the Principia*, I find to be Hyperbolas; therefore it will be sufficient to compute the Orbit of one Body only, as of *L*, whose transverse Axe is 24833,56, conjugate 218336,4, & *latus rectum* 1919612,2 Miles.





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V. The Orbit being constructed, as in Scheme the 3d, * and the Area computed, which the Body *L* describes in 48 Hours = 26242565760 Miles, and this Area lessened by the given Area *ACT*, gives the Area to be cut off equal 26022305521 Miles. Whence by *Prob. 23.* of the *Principia*, the Distance of the Body *L* from the Center is 503826 Miles, of *T* 12596 Miles, and the absolute Distance of the Bodies from each other 516422 Miles, and the Angle at the Center, or mean Anomaly of each Body is $84^{\circ} : 8'$.

VI. But the Center of Gravity, in 48 Hours, has moved forwards 403350 Miles; therefore the Body *T* makes with the Point of Projection *B* an Angle of $59^{\circ} : 55' : 28''$, and its Distance from it is 393270 Miles: Also *L* makes with *A* an Angle of $98^{\circ} : 23' : 15''$, and its Distance from it is 865170 Miles, the Answer.

* In Scheme the 3d, *AB* represents the Plane the Bodies were projected from, *A'B'* its Position when the Bodies have been 48 Hours in Motion, *TT* the transverse Axe of their Orbits, *A'B'* the Points the Bodies set out from when they begin to describe Hyperbolical Orbits round the common Center of Gravity, *C*; the pricked Line is the Path of the Center of Gravity, and the Line *LT* the Line connecting their Centers, when their Distances and Positions are required.

BT is the Distance of *T* from the Point of Projection, after it has been 48 Hours in Motion, *AL* the Distance of *L*, and the respective Angles *TBA* and *LAB* give their Position

The Scheme is not accurately drawn, it being impossible, there is such a Difference in the Length of the Lines that respect *L*.

QUES.

QUESTION III.

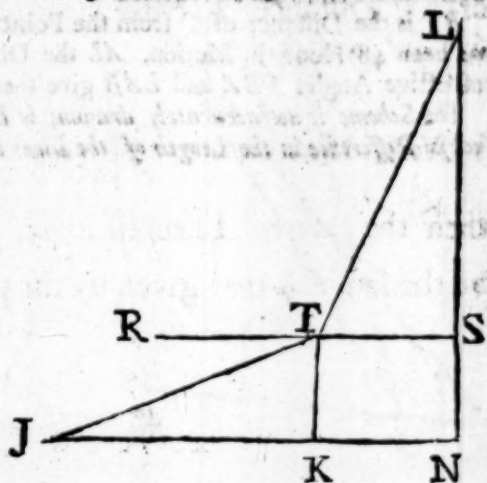
By Mr. NICH. OATS.

A Fleet of Ships at Portsmouth is bound with military Stores, &c. for our brave Admiral in the West-Indies; and being inform'd by experienced Navigators, that a Ship, in sailing upon a Wind, having her Larbourd Tacks on board, which makes her way good six Miles an Hour, will, when got into a Trade Wind (which blows in the Latitude of thirty) make her way nine Miles an Hour; now admit the Fleet can sail at the Rate above, I demand the Course and Distance before and after their Arrival in the Trade Wind, to be perform'd in the shortest Time possible, from the Lizard to Jamaica, and the Minimum according to Wright's Projection. Lizard in Lat. $49^{\circ} 56'$, Long. $5^{\circ} 14'$ West, Jamaica 18° and 76° .

SOLUTION by Mr. J. POWLE of Hereford.

IN the annexed Scheme *L* represents the Lizard, *RS* the Parallel of 30° , *T* a Point therein the Fleet must come to, in their Way to Jamaica, denoted by *J*, the Angle *TLS* the first Course steered, &c.

We have given (per Wright's Projection) $LS = 1579,8979$ Miles, which call *b*, SN or $TK = 790,1607 =$



$c, N\gamma = 4246 = d$, put $x = TS$, $m = 6$, $n = 9$;
 then per 47 1 $TL = \sqrt{bb + xx}$, $T\gamma = \sqrt{cc + dd - 2dx + x^2}$, and by a uniform Velocity the
 Times of Description will be $\frac{\sqrt{bb + x^2}}{m}$, and
 $\frac{\sqrt{cc + dd - 2dx + x^2}}{n}$; their Sum $\frac{\sqrt{bb + xx}}{m} + \frac{\sqrt{cc + dd - 2dx + x^2}}{n}$ must be a Min. Its Flux, if
 made $= 0$ and ordered, there will arise this Equation:

$$\begin{array}{r} +nn \left| \begin{array}{l} -2dn^2 \\ +2dm^2 \end{array} \right| x^4 \quad \begin{array}{l} +ddnn \\ +ccnn \\ -bbm^2 \\ -ddm^2 \end{array} \left| \begin{array}{l} x^3 \\ x^3 \end{array} \right| \quad \begin{array}{l} +2db^2m^2x = b^2d^2m^2. \end{array}$$

In Numbers, $54x^4 - 382140x^3 + 771997102,9985x^2 + 763080806290,03x = 1620020551753744$.

Solved, $x = 1458,016$, or $2891,56$, or $5488,45$, or $1346,03$. But it's the first of these Values ($1458,016$) which serves to our present Purpose. Now having found TS , by plain Trigonometry TL , the Distance sail'd before the Fleet's Arrival in the Trade Winds will be had $2149,86$; Course steer'd $SWbS$ $8^\circ : 27'$ westerly: Thence to *Jamaica*, or the Distance $T\gamma$, is $2897,79$; Course WSW $6^\circ : 40'$ westerly.

Note, If n be put $= \frac{9 \times 790}{720}$, and $m = \frac{6 \times 1580}{1196}$,

then the above Theorem, viz. $\frac{+nn}{-mm} \left| \begin{array}{l} x^4 \\ \&c. \end{array} \right|$ will be the same as that given by the proper.

Q U E S.

QUESTION IV.

By Mr. ROBERT HEATH.

*Four Maids wise and fair, and as Astrea rare,
 (Impatient to tie Hymen's Noose)
 Burn with amorous Flame for an Artist of Fame:
 Then which for her Age would you choose?*

*Supposing their several Ages to be represented by the
 following Equations.*

$$eea + uua + yya, = 20850.$$

$$aac + uue + yye, = 23238.$$

$$aa u + ee u + yy u, = 24654.$$

$$aay + eey + uuy, = 24750.$$

*What was each of their Ages, and the analytical
 Investigation?*

SOLUTION.

MR. Wright, Teacher of the Mathematicks
 at Pontefract, has given us the Analytical
 Investigation of this Problem, and finds $a=15$,
 $e=18$, $u=21$, and $y=25$, his Process at large will
 be inserted in the 3d Volume.



QUES.

QUESTION VI.

By F. R. S.

Let there be the Frustrum of a Cone, whose less Diameter is 20 Inches, its greater 40, and Length 90; which being cut by a Plane diagonally through the contrary Extremities of its two Diameters, will divide it into two Parts (called Hoofs) a greater and less; required a scientific Theorem for finding the solid Content of each Part.

By Mr. POWLE of Hereford.

MR. Shirtcliff in his Treatise on Gauging, p. 172. has given a General Theorem for the Hoof of any Pyramidical Frustrum; whether it be the Elliptical, Parabolical, or Hyperbolical; from which ingenious Treatise I have transcribed the two following Theorems for the Elliptical Hoofs;

Let D = the Diameter of the great Base, d = that of the lesser, b = Frustrum's Altitude; then

Theorem 1, $\overline{D^3 - d\sqrt{Da}} \times \frac{.2618Db}{D-d}$ the greater Hoof.

2. $\overline{D\sqrt{Da} - d^2} \times \frac{.2618db}{D-d}$ the lesser Hoof.

Making use of the Numbers given, 48439,115 is the Solidity of one Hoof, 17125,815 that of the other.

Dd

Q U E S

QUESTION VII.

By Mr. WILLIAM DANIEL.

It is universally agreed, that the Heat of any Moment of Time, on any Day, is proportionable to a Rectangle made of the Sine of the Sun's Altitude, and the Arch of Time expressing his Continuance above the Horizon; which being allow'd, it is required to find what Time of the Day will be the hottest at Coventry (Latitude $52^{\circ}:30'$) on August 14, 1743.

SOLUTION by Mr. J. WATTS of Comb.

THEOREM. By the Data it is the hottest Time of the Day, when the Sun's Altitude divided by the Arc of Time expressing his Continuance above the Horizon, is equal to a Rectangle of the Sine of the comp. Latitude, comp. of the Sun's dec. and Sine of the Hour from Noon; which in Latitude $52^{\circ}:30'$, and on August 14. is 55 Minutes and 38 Seconds past One o'Clock.
Q. E. I.



QUES.

QUESTION VIII.

By Mr. JOHN MAY, junior, of Amsterdam.

Last Spring, 1742. being at Sea in North Latitude, we had great Storms for several Days together, succeeded with cloudy Weather, which hindered our making Observations; at last, when it cleared up, with Mr. Hadley's Octant upon Deck, I endeavoured to take the Sun's Meridian Altitude, but unfortunately thick Clouds obstructed it. Now being at a great Loss to know where we were, we endeavoured to contrive some other Way; and accordingly waiting a few Minutes, it cleared up again, and we took the Sun's Altitude (after Allowance for Refraction, and Dip of Horizon, &c.) $57^{\circ} : 24' : 52''$. Tarrying 26 Minutes by a good Watch, we observed the Sun's Height $55^{\circ} : 35' : 19''$; and 25 Minutes after this, the Sun's Altitude was $53 : 16' : 15''$. From whence is required the Latitude we were in, the Time of each Observation, and the Sun's Declination, by a general Theorem for all Problems of this Nature?

SOLUTION by Mr. J. BETTS of Oxford.

PUT *a*, for the Difference betwixt the Sines of the Sun's Altitude, 1st and 3^d Time observed; *b* for the Difference betwixt the Sines of the Sun's Altitude

Altitude 2d and 3d Time observed; s and v for the right and versed Sine of (25 Minutes $=$) $6^{\circ} : 15'$, S and V for the right and versed Sine of (26 Minutes $+ 25$ Minutes $=$) $12^{\circ} : 45'$ Radius $= 1$;

then will $\frac{bV - av}{vS - as} = .4684039$ the Tangent of the

Hour-Angle from Noon at the 3d Observation; which corresponds to $25^{\circ} : 6'$, whose Cosine call c , and put $d =$ Cosine of $12^{\circ} : 21'$, the Arch from Noon when the 1st Observation was made, and let $n =$ Sine of the Sun's Altitude at the 1st Observation; then will the Sine of the Sun's Meridian Altitude be $\frac{1 - d \times a}{d - c} + n = .8559346 = 58^{\circ} : 52' = r$,

of his Depression at Midnight $\frac{2a}{d - c} = .8559346$.

$= .2975683 = 17^{\circ} : 18' = s$; whence $90 - \frac{r + s}{2} =$

$51^{\circ} : 55'$ the Latitude of the Place, and $\frac{r - s}{2} =$

$20^{\circ} : 47'$ the Sun's Declination answering to the 13th of May.

N. B. *Whoever would find the Latitude of a Place exactly, cannot possibly meet with a better Theorem, to effect the same, than that which is given above.*



Q U E S

QUESTION IX.

By Mr. JOHN POWLE of Hereford.

Let there be three spherical, perfectly elastic Bodies, A, B, C: The Weight of A = 3 Pound, C = 27. Now it is required to find the Weight of the intermediate Body B; so that A striking B at Rest, and B with the Motion acquired by the Stroke, striking C at Rest, the Motion produced in C shall be a Maximum?

SOLUTION by Mr. BETTS of Oxford.

LET v , be any Velocity whereby the Body A is impelled, then by Mechanics $A+B:2A::v:$
 $\frac{2Av}{A+B}$ the Velocity of b after the Stroke; again,
 $B+C:2B::\frac{2Av}{A+B}:\frac{4ABv}{A+B \times B+C}$ the Velocity of
 C after the Stroke, a Maximum per Question; $\therefore$
 $ACB - BB^2 = 0$, or $AC = B^2 \therefore \sqrt{AC} = B = 9$,
 Q. E. I.

QUEE

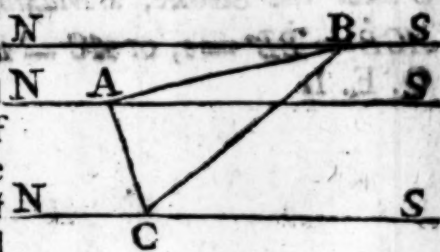
QUESTION X.

By the late illustrious Sir ISAAC NEWTON.

Three Staves being erected, or set up on end, in some certain Place of the Earth, perpendicular to the Plane of the Horizon, in the Points A, B, and C; whereof that which is at A, is 6 Foot long; that in B, 18; that in C, 8; the Line AB being 30 Foot long: It happens on a certain Day in the Year, that the End of the Shadow of the Staff A, passes through the Points B and C; and of the Staff B, through A and C; and of the Staff C, through the Point A. To find the Sun's Declination, and the Elevation of the Pole, or Day and Place where this shall happen?

SOLUTION.

CALL the Height of the Staff placed in the Point A, (a) that in B, (b) that in C (c) the Line AB (d) the Distance betwixt the Top of the Staff in A, and the Bottom of the Staff in B, (e) the Distance betwixt the Top of the Staff in B, and Bottom of the Staff in A, (f) and let $AC = x$, $CB = z$; then per 47. e. 1. $\sqrt{xx+aa} = s$, and $\sqrt{x^2+c^2} (=m)$ = the Distances betwixt the Top of the Staff in A and Bottom of the Staff in C,



C , and Top of the Staff in C and Bottom of the Staff in A , respectively. And suppose NS the Meridian Line, Radius $= 1$; then will $\frac{a}{e}$ and $\frac{d}{e}$ be the Sine and Cosine of the Sun's Altitude when he makes the Shade AB by the Staff A ; for the Cosine of the Angle ABN (or BAN) = Cosine of the Sun's Azimuth from the North at that Time, put $\mp v$. Again, $\frac{b}{f}$ and $\frac{d}{f}$ expresses the Sine and Cosine of the Sun's Altitude when he makes the Shade BA by the Staff B ; the Cosine of BAN (or ABN) = Cosine of the Sun's Azimuth from the North at that Time $\mp v$.

Also $\frac{a}{s}$ and $\frac{x}{s}$ will be the Sine and Cosine of the Sun's Altitude when he makes the Shade AC by the Staff A ; and let $\mp y$, be put to denote the Cosine of the Angle ACN (or CAN) = the Sun's Azimuth from the North at that Time; again, $\frac{c}{m}$ and

$\frac{x}{m}$ will express the Sine and Cosine of the Sun's Altitude when he makes the Shade CA , by the Staff C ; and $\mp y$, the Cosine of the Angle CAN (or ACN) = the Cosine of the Sun's Azimuth at that Time from the North. Lastly, Let p and q be put to express the Sine and Cosine of the required Latitude, and g , the Sine of the Sun's Declination; then per Theorem the 8th, p. 99. we have (1.)

$$\frac{pa}{e} \mp \frac{qdv}{e} = g; (2.) \frac{pb}{f} \mp \frac{qdv}{f} = g; (3.) \frac{pa}{s} \mp \frac{qxy}{s} = g (4.) \frac{pc}{m} \mp \frac{qxy}{m} = g. \text{ Whence per 1st Equation we have } \pm qdv = ge - pa; \text{ and per 2d}$$

$pb =$

$pb - gf = + qdv$; consequently $gc - pa = pb - gf$, and

$g = \frac{pb + pa}{c + f}$; again, per 3d Equation we get $+$

$qxy = gs - pa$, and per 4th $pc - gm = + qxy$:

Consequently $gs - pa = pc - gm$ $\therefore g = \frac{pc + pa}{s + m}$

therefore $\frac{pc + pa}{s + m} = \frac{pb + pa}{c + f}$, and $\frac{c + a}{s + m} = \frac{b + a}{c + f}$

out of Fractions, &c. its $\frac{c + a \times c + f}{b + a} = s +$

$m \therefore \frac{c + a \times c + f}{b + a} - m = s$, which squared is

$$\frac{(c + a)^2 \times c + f^2}{(b + a)^2} - \frac{2m \times c + a \times c + f}{b + a} + m^2 = s^2$$

but as $\sqrt{x^2 + a^2} = s$, and $\sqrt{x^2 + c^2} = m$ $\therefore (x^2 =)$

$s^2 - a^2 = m^2 - c^2 \therefore s^2 = m^2 + a^2 - c^2$, which put

for s^2 in the Equation above, makes $\frac{(c + a)^2 \times c + f^2}{(b + a)^2}$

$- \frac{2m \times c + a \times c + f}{b + a} + m^2 = m^2 + a^2 - c^2 \therefore$ by

Transposition, &c. we have $\frac{c + a \times c + f}{2b + 2a} + \frac{b + a \times c - a}{2c + 2f}$

$= m$ the Distance betwixt the Top of the Staff in C and Bottom of the Staff in A , and so AC is found. And by the very same Way that m was found, the Distance betwixt the Top of the Staff in B and Bottom of the Staff in C may be found, and

if n be put for it, the Equation will stand $\frac{c + b \times c + f}{2b + 2a}$

$+ \frac{b - c \times b + a}{2c + 2f} = n$, and therefore CB is known.

For the other Part of the Analysis, we have by the Help

of 1st and 2d Equations $\frac{pa}{e} \pm \frac{qdv}{e} = \frac{pb}{f} \mp \frac{qdv}{f} \therefore$

$$\pm \frac{qdv}{e} \pm \frac{qdv}{f} = \frac{pb}{f} - \frac{pa}{e} \therefore \pm v = \frac{be - fa}{df + de} \times \frac{p}{q}$$

put $\frac{be - fa}{df + de} = b$, then will $\pm v = b \frac{p}{q}$ whose Square

is $v^2 = b^2 \frac{p^2}{q^2} \therefore \sqrt{1 - v^2} = \sqrt{1 - b^2 \frac{p^2}{q^2}}$ the Sine of the Angle *BAS*.

And by the same Way of Reasoning, from the 3d and 4th Equations, is found $\pm y = \frac{cs - am}{xm + xs} \times \frac{p}{q}$

Put $\frac{cs - am}{xm + xs} = k$ then will $\pm y = k \frac{p}{q}$, which

squared makes $y^2 = k^2 \frac{p^2}{q^2} \therefore \sqrt{1 - y^2} = \sqrt{1 - k^2 \frac{p^2}{q^2}}$

the Sine of the Angle *CAS*.

Now having all the three Sides of the Triangle *ABC* given, we find the Angle *CAB* to be less than a right Angle; therefore the Cosine belonging

to the Sum of the Angles *BAS* and *CAS* is $bk \frac{p^2}{q^2}$

$-\sqrt{1 - b^2 \frac{p^2}{q^2}} \sqrt{1 - k^2 \frac{p^2}{q^2}}$ (see Theorem 2. p. 3.)

But *per* Nature of the Triangle we have $\frac{x^2 + d^2 - z^2}{2dx}$

= *v* the Cosine of the Angle *CAB*, consequently

$$bk \frac{p^2}{q^2} - \sqrt{1 - b^2 \frac{p^2}{q^2}} \sqrt{1 - k^2 \frac{p^2}{q^2}} = v \therefore bk \frac{p^2}{q^2} - v =$$

$\sqrt{1 - b^2 \frac{p^2}{q^2}} \sqrt{1 - k^2 \frac{p^2}{q^2}}$ which squared, &c. makes

$$-2bkv \frac{p^2}{q^2} + v^2 = 1 - b^2 \frac{p^2}{q^2} - k^2 \frac{p^2}{q^2}, \text{ or by writing}$$

E c

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ing $1 - p^2$ for q^2 its $-\frac{2bkvp^2}{1-p^2} + v^2 = 1$

$\frac{b^2p^2}{1-p^2} - \frac{k^2p^2}{1-p^2}$ And by Reduction $2p^2 =$

$\frac{2-2v^2}{1+b^2+k^2-2bkv-v^2}$ the versed Sine of an Arc,

which is double the required Latitude. And the Sine of the Sun's Declination is at p. 208. l. 4. known

to be $g = \frac{pc + pa}{s + m}$. Q. E. I.

The END of the First Volume.

